1 L'Hospital's Rule and Indeterminate forms

The name of the game with sequences is to determine whether the a_n approaches a limit L as $n \to \infty$ and if possible, to determine what that limit might be. We have seen a few tools so far:

- Direct calculation, either from limit rules and known limits of related sequences, or from the definition of limit itself.
- The theorem on bounded monotonic sequences. We saw this was useful with some recursively defined sequences, but one needed to know the LUB or GLB in order to actually find the limit.
- The Squeeze Theorem. One needs to be lucky or clever enough to find bounding sequences above and below the subject sequence which converge to the same limit.
- Comparison with a function $f(x)$ with the property that $f(n) = a_n$. This is probably the most natural thing to try. However, many sequences of interest are in the form of a quotient, and many of these are indeterminate of form $0/0$ or ∞/∞ as we look at the limit. L'Hospital's Rule gives us a way of finding the limits when we have indeterminate forms of these types.

In order to prove this rule we should go back and review several mean-value type theorems.

Rolle's Theorem Let f be differentiable on (a, b) and continuous on $[a, b]$ and suppose that $f(a) + f(b) = 0$. Then there is at least one $c \in (a, b)$ with the property that $f'(c) = 0$.

The Mean Value Theorem Let f be differentiable on (a, b) and continuous on $[a, b]$. Then there is at least one $c \in (a, b)$ with the property that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

You can think of Rolle's theorem as a special case of the Mean Value Theorem with $g(x) = x$.

The Cauchy Mean Value Theorem Let f and g be differentiable on (a, b) and continuous on $[a, b]$, and further suppose that $g'(x)$ is never 0 in (a, b). Then there is at least one point c in (a, b) such that

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$

This can be proved by applying Rolle's Therem to the function

$$G(x) = [g(b) - g(a)][f(x) - f(a)] - [g(x) - g(a)][f(b) - f(a)]$$

We use this result to prove

L'Hospital's Rule (0/0)

If f and g are differentiable functions with $g'(x) \neq 0$ and if f and g both approach 0 as $x \to c$ (c may be finite or infinite, and the limit may be one-sided or two-sided), then if

$$\lim x \to c \frac{f'(x)}{g'(x)} = L, \text{ then } \lim x \to c \frac{f(x)}{g(x)} = L$$

If L is infinite, the result still holds.

The book does a proof of the one-sided case (from the right) for a finite c and also the case $x \to \infty$. The case for the one-sided limit from the left and $x \to -\infty$ are similar.

Examples: I'll use the symbol $\stackrel{\text{L'H}}{=}$ to remind me that the = sign is courtesy of L'Hospital's Rule. Also, note that if the result from the application of the rule is again indeterminate, we may apply the rule again until we arrive at a resolution of the indeterminacy. These applications may be daisy-chained together to give the limit for the original expression.

1. $\lim x \to 4 \dfrac{x^2 - x - 12}{x^2 - 3x - 4} \stackrel{\text{L'H}}{=} \lim_{x \to 4} \dfrac{2x - 1}{2x - 3} = 7/5$

2. $\lim_{x \to 0} \dfrac{\sin x}{x} \stackrel{\text{L'H}}{=} \lim_{x \to 0} \dfrac{\cos x}{1} = 1$

3. $\lim_{x \to 0} \dfrac{x}{1 - e^x} \stackrel{\text{L'H}}{=} \lim_{x \to 0} \dfrac{1}{-e^x} = -1$

4. $\lim\limits_{x\to 1} \dfrac{1-x+\ln x}{x^3-3x+2} \overset{\text{L'H}}{=} \lim\limits_{x\to 1} \dfrac{-1+\frac{1}{x}}{3x^2-3} \overset{\text{L'H}}{=} \lim\limits_{x\to 1} \dfrac{\frac{-1}{x^2}}{6x} = -1/6$

5. $\lim\limits_{x\to\infty} \dfrac{\sin\left(\frac{1}{x}\right)}{\tan^{-1}\left(\frac{1}{x}\right)} \overset{\text{L'H}}{=} \lim\limits_{x\to\infty} \dfrac{\cos\left(\frac{1}{x}\right)\cdot\left(\frac{-1}{x^2}\right)}{\frac{1}{1+\frac{1}{x^2}}\left(\frac{-1}{x^2}\right)} \overset{\text{L'H}}{=} \lim\limits_{x\to\infty} \dfrac{\cos\left(\frac{1}{x}\right)}{\frac{x^2}{1+\frac{1}{x^2}}} = 1$

It turns out that L'Hospital's Rule is also valid for the indeterminate form ∞/∞. Here are some example calculations:

1. $\lim\limits_{x\to 0^+} \dfrac{\ln x}{\frac{1}{x}} \overset{\text{L'H}}{=} \lim\limits_{x\to 0^+} \dfrac{\frac{1}{x}}{\frac{-1}{x^2}} = \lim\limits_{x\to 0^+} (-x) = 0$

2. $\lim\limits_{x\to\infty} \dfrac{x^2}{e^x} \lim\limits_{x\to\infty} \dfrac{2x}{e^x} = \lim\limits_{x\to\infty} \dfrac{2}{e^x} = 0$

3. $\lim\limits_{x\to\frac{\pi}{2}} \dfrac{\sec^2 x}{\sec^2 3x}$

   You might attempt doing this by applying L'Hospital's Rule to this expression, but may find that it just keeps getting more complicated. In this case it might be best first to express the thing in terms of sines and cosines, since we are more familiar with these.

$$\begin{aligned}
\lim_{x\to\frac{\pi}{2}} \frac{\sec^2 x}{\sec^2 3x} &= \lim_{x\to\frac{\pi}{2}} \frac{\cos^2 3x}{\cos^2 x} \\
&\overset{\text{L'H}}{=} \lim_{x\to\frac{\pi}{2}} \frac{-6\cos 3x\sin 3x}{-2\cos x\sin x} \\
&= \lim_{x\to\frac{\pi}{2}} \frac{3(2\cos 3x\sin 3x)}{2\cos x\sin x} \\
&= \lim_{x\to\frac{\pi}{2}} \frac{3\sin 6x}{\sin 2x} \\
&\overset{\text{L'H}}{=} \lim_{x\to\frac{\pi}{2}} \frac{18\cos 6x}{2\cos 2x} = \frac{18(-1)}{2(-1)} = 9
\end{aligned}$$

Another indeterminate form which arises a lot is the form $0\cdot\infty$. Here we can easily transform the expression into one of the above types by taking the reciprocal of one of the functions and placing it in the denominator.

Example: $\lim\limits_{x\to 0} \sin^{-1} x \csc x = \lim_{x\to 0} \frac{\sin^{-1} x}{\sin x} \overset{\text{L'H}}{=} \lim_{x\to 0} \frac{\frac{1}{\sqrt{1-x^2}}}{\cos x} = 1$

Example: $\lim\limits_{x\to\infty} 5(xe^x - 1)$

Here it seems best to leave the more complicated expression in the numberator:

$$\lim_{x\to\infty} 5(xe^{1/x} - 1) = 5 \lim_{x\to\infty} \frac{e^{1/x} - 1}{1/x} \overset{\text{L'H}}{=} 5 \lim_{x\to\infty} \frac{e^{1/x}\left(\frac{-1}{x^2}\right)}{\frac{-1}{x^2}} = 5 \lim_{x\to\infty} e^{1/x} = 5$$

We also have the form $\infty - \infty$ which needs to be adjusted algebraically, perhaps by getting a common denominator or using some rationalizing trick, to be expressed as a fraction.

Example: $\lim\limits_{x\to 0} \left(\frac{1}{x^2} - \frac{1}{x^2 \sec x}\right)$

Here we can do a little algebra:

$$\lim_{x\to 0} \left(\frac{1}{x^2} - \frac{1}{x^2 \sec x}\right) = \lim_{x\to 0} \left(\frac{1}{x^2} - \frac{\cos x}{x^2}\right) = \lim_{x\to 0} \frac{1 - \cos x}{x^2}$$

We now apply L'Hospital's Rule:

$$\lim_{x\to 0} \frac{1 - \cos x}{x^2} \overset{\text{L'H}}{=} \lim_{x\to 0} \frac{\sin x}{2x} \overset{\text{L'H}}{=} \lim_{x\to 0} \frac{\cos x}{2} = 1/2$$

Example: $\lim\limits_{x\to\infty} \sqrt{x^2 + 5x - 20} - x$

Here we multiply by 1 to get a quotient:

$$\lim_{x\to\infty} \sqrt{x^2 + 5x - 20} - x = \lim_{x\to\infty} \sqrt{x^2 + 5x - 20} - x \frac{\sqrt{x^2 + 5x - 20} + x}{\sqrt{x^2 + 5x - 20} + x} = \lim_{x\to\infty} \frac{x^2 + 5x + 20 - x^2}{\sqrt{x^2 + 5x - 20} + x}$$

Now applying L'Hospital's Rule since this is of the form (∞/∞),

$$\lim_{x\to\infty} \frac{5x + 20}{\sqrt{x^2 + 5x - 20} + x} \overset{\text{L'H}}{=} \lim_{x\to\infty} \frac{5}{\frac{1}{2}(x^2 + 5x + 20)^{-1/2}(2x + 5) + 1}$$

We see that the numerator goes to 5; we need to figure out what the denominator is going to. Dropping the 1 for the moment,

$$\begin{aligned}
\lim_{x\to\infty}\frac{1}{2}(x^2+5x+20)^{-1/2}(2x+5) &= \lim_{x\to\infty}x(x^2+5x+20)^{-1/2}+\lim_{x\to\infty}\frac{5}{2}(x^2+5x+20)^{-1/2} \\
&= \lim_{x\to\infty}x(x^2+5x+20)^{-1/2}+0
\end{aligned}$$

But,

$$\lim_{x\to\infty}x(x^2+5x+20)^{-1/2}=\lim_{x\to\infty}\frac{x}{\sqrt{x^2+5x+20}}=\lim_{x\to\infty}\frac{1}{\sqrt{1+5/x+20/x^2}}=1$$

So, we have

$$\lim_{x\to\infty}\frac{5}{\frac{1}{2}(x^2+5x+20)^{-1/2}(2x+5)+1}=\frac{5}{1+1}=5/2$$

Other indeterminate forms are exponential in nature: $0^0, \infty^0$ and 1^∞ are all indeterminate forms. Here we use the trick of taking the log of the expression first, and then computing the limit. You can see that any of these expressions when logified will result in something of the indeterminate form $0\cdot\infty$ which we now know how to handle. When we find the limit of the logified expression, we have to remember to apply e^x to this limit to recover the limit of the original expression.

Example: $\lim\limits_{x\to 0}(x+1)^{\cot x}$

Here we let $y=(x+1)^{\cot x}$ so that $\ln y=\cot x\ln(x+1)$. Rewriting this so we can apply L'Hospital's Rule, we get

$$\lim_{x\to 0}\ln y=\lim_{x\to 0}\frac{\ln(x+1)}{\tan x}\overset{\text{L'H}}{=}\lim_{x\to 0}\frac{\frac{1}{x+1}}{\sec^2 x}=1$$

Since $\ln y\to 1$ we know that $y\to e$; thus, $\lim_{x\to 0}(x+1)^{\cot x}=e$.