Math114 – Linear Algebra

Essentials update

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- VP is at the *help desk:* Tuesday July 25, 9-11 am; Friday July 28, 9-11 am. (Consultation room Cotton 436). + other people at the help desk, at other times
- *Tutorials:* start this week.
- *Transparencies:* go to the course page `http://www.mcs.vuw.ac.nz/courses/MATH114` in .ps and .pdf format.
- Recommended reading: Anton, *Elementary Linear Algebra,* Ch. I and II.

Lecture 3

We shall write $\vec{a}$, $\vec{b}$, ... or $\mathbf{a}$, $\mathbf{b}$, ..., for row vectors, and $\vec{a}^T$, $\vec{b}^T$, ... or $\mathbf{a}^T$, $\mathbf{b}^T$, ... for column vectors.

ex.: if $\mathbf{x} = \vec{x} = (x_1, x_2, x_3)$, then
$$\mathbf{x}^T = \vec{x}^T = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

♣ *Plain letters are used for scalars (= ordinary numbers)*

In writing: use $\underset{\sim}{x}$ instead of $\mathbf{x}$ or $\vec{x}$.

Note: $\left(A^T\right)^T = A$.

(reflect A in its diagonal twice, and you get A back again)

(Those things we were doing to matrices when solving linear eq's:)

Elementary Row Operations

1) Multiply throughout a row by a non-zero scalar
2) Interchange two rows (we haven't had yet to do this)
3) To a row, add a scalar multiple of another row.

(Anton, p. 5)

Elementary row operations can change a matrix a lot, e.g. $\begin{pmatrix} 4 & 2 & 4 \\ 6 & 8 & 1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 0 & \frac{3}{2} \\ 0 & 1 & -1 \end{pmatrix}$ in 4 EROs.

def. A matrix A is *row equivalent* (*RE*) to B if some finite sequence of elementary row operations changes A to B.

notation: $A \overset{RE}{\cong} B$.

♣ this is in fact an equivalence relation: reflexive, symmetric, and transitive.

♣ similarly, one can define *elementary column operations* and the *column equivalent* matrices ($A \overset{CE}{\cong} B$); we will not be using these.

warning: beware of trying to perform too many EROs at once, to avoid errors.

What is the *object* of these changes?
We aim at a matrix of this type:

$$\begin{pmatrix} 1 & & & & \text{Other things} \\ & 1 & & & \text{here} \\ & & 1 & & \\ & & & 1 & \\ \text{0's} & \text{here} & & & \end{pmatrix}$$

(a 'staircase pattern'). More exactly:

def. A matrix A is in the *reduced row-echelon form* if it has the following properties:

- if a row does not consist entirely of zeros, then the first nonzero entry in it is 1. ('a leading 1')
- any rows that consist entirely of zeros are grouped together at the bottom.
- the leading 1 in the lower row always occurs farther to the right than the leading 1 in the higher row
- each column that contains a leading 1 has zeros everywhere else.

ex.: all matrices obtained by Gauss–Jordan elimination in the previous lectures were in the reduced row-echelon form.

Gauss–Jordan elimination =
putting the coefficient matrix of a system of linear equations into the reduced row-echelon form by means of elementary row operations.

A bigger **ex.** of solving linear eqns (other ex.: Anton, §1.2)

$$\begin{array}{rcrcrcr} 2x & + & 6y & + & 6z & = & 2 \\ 3x & + & 9y & - & 3z & = & -5 \\ -2x & - & 3y & - & 3z & = & 3 \\ x & - & y & + & 12z & = & 3 \end{array}$$

$$\left(\begin{array}{rrr:r} \underline{2} & 6 & 6 & 2 \\ 3 & 9 & -3 & -5 \\ -2 & -3 & -3 & 3 \\ 1 & -1 & 12 & 3 \end{array}\right)$$

$$\underset{r_1\times\frac{1}{2}}{\overset{RE}{\cong}} \begin{pmatrix} 1 & 3 & 3 & 1 \\ \underline{3} & 9 & -3 & -5 \\ \underline{-2} & -3 & -3 & 3 \\ \underline{1} & -1 & 12 & 3 \end{pmatrix}$$

$$\underset{\substack{r_2-3r_1 \\ r_3+2r_1 \\ r_4-r_1}}{\overset{RE}{\cong}} \begin{pmatrix} 1 & 3 & 3 & 1 \\ 0 & 0 & -12 & -8 \\ 0 & 3 & 3 & 5 \\ 0 & -4 & 9 & 2 \end{pmatrix}$$

we have tidied up the 1st column: $\Rightarrow x$ is only left in the 1st eqn.

How to maintain the staircase pattern in the 2nd column? The 0 entry is no good! $\Rightarrow$ interchange rows:

$$\underset{r_2\leftrightarrow r_3}{\overset{RE}{\cong}} \begin{pmatrix} 1 & 3 & 3 & 1 \\ 0 & \underline{3} & 3 & 5 \\ 0 & 0 & -12 & -8 \\ 0 & -4 & 9 & 2 \end{pmatrix}$$

now we can get a $1 = A_{22}$ by division

$$\underset{r_2\times\frac{1}{3}}{\overset{RE}{\cong}} \begin{pmatrix} 1 & 3 & 3 & 1 \\ 0 & 1 & 1 & \frac{5}{3} \\ 0 & 0 & -12 & -8 \\ 0 & \underline{-4} & 9 & 2 \end{pmatrix}$$

use that 1 to change everything *below* it to 0

(never mind what is left *above*! $\rightsquigarrow$ later).

$$\underset{r_4+4r_2}{\overset{RE}{\cong}} \begin{pmatrix} 1 & 3 & 3 & 1 \\ 0 & 1 & 1 & \frac{5}{3} \\ 0 & 0 & \underline{-12} & -8 \\ 0 & 0 & 13 & \frac{26}{3} \end{pmatrix}$$

$$\underset{r_3\times\left(-\frac{1}{12}\right)}{\overset{RE}{\cong}} \begin{pmatrix} 1 & 3 & 3 & 1 \\ 0 & 1 & 1 & \frac{5}{3} \\ 0 & 0 & 1 & \frac{2}{3} \\ 0 & 0 & \underline{13} & \frac{26}{3} \end{pmatrix}$$

$$\underset{r_4-13r_3}{\overset{RE}{\cong}} \begin{pmatrix} 1 & 3 & 3 & 1 \\ 0 & 1 & 1 & \frac{5}{3} \\ 0 & 0 & 1 & \frac{2}{3} \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The *Gauss* elimination is over. The so-called *row-echelon form* (all properties but 4).
Now use leading 1s to change everything above them to 0, *working from right to left*.

Start with the lowest leading 1.

$$\underset{\substack{r_1-3r_3 \\ r_2-r_3}}{\overset{RE}{\cong}} \begin{pmatrix} 1 & 3 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & \frac{2}{3} \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

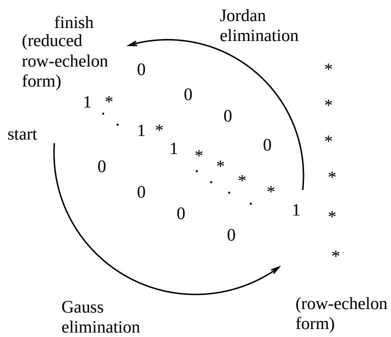
$$\underset{r_1-3r_2}{\overset{RE}{\cong}} \left(\begin{array}{rrr:r} 1 & 0 & 0 & -4 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & \frac{2}{3} \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Reduced row-echelon form. (Check!)
And that's it:

$$\begin{array}{rcr} x & = & -4 \\ y & = & 1 \\ z & = & \frac{2}{3} \end{array}$$

The 4th row tells us nothing about $x, y, z \Rightarrow$ forget it.
Unique solution: $(x, y, z) = \left(-4, 1, \frac{2}{3}\right)$.

Gauss–Jordan elimination procedure:



♣ x occurs in at most one equation,
y occurs in at most one eq., ...