DETERMINANTS AND CRAMER'S RULE

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From elementary algebra one is familiar with the concept of a determinant which arises when we find solutions of simultaneous equations in two or three independent variables.

Example 1. Find the solution of the three simultaneous equations:

$$\begin{aligned} 2x + 4y + z &= b_1, \\ 4x + 2y - z &= b_2, \\ 7x + y - 3z &= b_3, \end{aligned}$$

where x, y and z are the three unknowns and b_1, b_2 and b_3 are the three known quantities. Determinants also arise when we find 2-dimensional area, 3-dimensional or in general an n-dimensional volume, equation of a line passing through three points, etc., etc.

Example 2. Find the vector area subtended by the two vectors

$$\mathbf{V}_1 = \{2, 6\}\ ,\ \mathbf{V}_2 = \{1, 4\}; \tag{1}$$

or, find the scalar volume subtended by the three vectors

$$\mathbf{V}_1 = \{2, 6, 5\}\ ,\ \mathbf{V}_2 = \{3, 1, 4\}\ ,\ \mathbf{V}_3 = \{1, 2, 4\}. \tag{2}$$

Determinants also arise when dealing with resistive circuits, where to find the flow of currents in various resistive branches of the circuit we have to use Cramer's rule. Determinants also arise when we deal with a system of first-order electric circuits.

Consider an $(n \times n)$-matrix $\mathbf{A}$ as shown below:

$$\mathbf{A} = \begin{bmatrix} a_1^1 & a_1^2 & a_1^3 & \ldots & \ldots & a_1^n \\ a_2^1 & a_2^2 & a_2^3 & \ldots & \ldots & a_2^n \\ \vdots & \vdots & \vdots & & & \vdots \\ a_n^1 & a_n^2 & a_n^3 & \ldots & \ldots & a_n^n \end{bmatrix}. \tag{3}$$

Associated with this square matrix is a number called the *determinant* and we write it as:

$$|\mathbf{A}| = \begin{vmatrix} a_1^1 & a_1^2 & a_1^3 & \ldots & \ldots & a_1^n \\ a_2^1 & a_2^2 & a_2^3 & \ldots & \ldots & a_2^n \\ \vdots & \vdots & \vdots & & & \vdots \\ a_n^1 & a_n^2 & a_n^3 & \ldots & \ldots & a_n^n \end{vmatrix}. \tag{4}$$

We call $|\mathbf{A}|$ the determinant of matrix $[\mathbf{A}]$. Sometimes it is also written as $\text{Det}(\mathbf{A})$ or Δ (where $\Delta \equiv$ Delta for D in Greek).

In the notation used here the subscript denotes the row and superscript denotes the column. Thus, a_4^3 is located in the 4-th row of the 3-rd column. In general the element a_n^m is located at the intersection of the

n-th row and m-th column. An element lying on the principal diagonal is denoted by a_m^m (no summation intended). Thus the element a_3^3 in a (4×4)-determinant is a diagonal element lying on the principal diagonal at the intersection of 3-rd row and 3-rd column.

1 MINORS

Consider an $(n \times n)$-order matrix $\mathbf{A}$. Its determinant is

$$|\mathbf{A}| = \left| \begin{array}{cc|cccc|cc} a_1^1 & a_1^2 & a_1^3 & a_1^4 & \dots & a_1^m & \dots & a_1^n \\ \cline{3-6} a_2^1 & a_2^2 & a_2^3 & a_2^4 & \dots & a_2^m & \dots & a_2^n \\ a_3^1 & a_3^2 & a_3^3 & a_3^4 & \dots & a_3^m & \dots & a_3^n \\ a_4^1 & a_4^2 & a_4^3 & a_4^4 & \dots & a_4^m & \dots & a_4^n \\ . & . & . & . & . & . & . & . \\ a_m^1 & a_m^2 & a_m^3 & a_m^4 & \dots & a_m^m & \dots & a_m^n \\ \cline{3-6} . & . & . & . & . & . & . & . \\ \vdots & \vdots & \vdots & & & & & \vdots \\ a_n^1 & a_n^2 & a_n^3 & a_n^4 & \dots & a_n^m & \dots & a_n^n \end{array} \right|_{(n \times n)} . \tag{5}$$

Now choose a number $m < n$. Consider now the $m \times m = m^2$ elements of the minor determinant $|\mathbf{M}|$, whose elements are common to m rows and m columns. Such an $(m \times m)$-determinant is called the minor determinant $|\mathbf{M}|$, or the m-th order minor of $|\mathbf{A}|$. Such an $(m \times m)$-minor is delineated by the dashed lines in the figure above. More specifically consider the minor $\mathbf{M}_{345}^{234}$ which consists of elements common to the 3-rd, 4-th, 5-th row and 2-nd, 3-rd, 4-th column; i.e.,

$$\mathbf{M}_{345}^{234} = \begin{bmatrix} a_3^2 & a_3^3 & a_3^4 \\ a_4^2 & a_4^3 & a_4^4 \\ a_5^2 & a_5^3 & a_5^4 \end{bmatrix} . \tag{6}$$

If we delete all the elements in the 3-rd, 4-th, 5-th row and in the 2-nd, 3-rd, 4-th column, we obtain the complementary minor

$$\overline{\mathbf{M}}_{126\dots n}^{156\dots n} = \begin{bmatrix} a_1^1 & a_1^5 & a_1^6 & \dots & a_1^n \\ a_2^1 & a_2^5 & a_2^6 & \dots & a_2^n \\ a_6^1 & a_6^5 & a_6^6 & \dots & a_6^n \\ \vdots & \vdots & \vdots & & \vdots \\ a_n^1 & a_n^5 & a_n^6 & \dots & a_n^n \end{bmatrix} . \tag{7}$$

The co-factor of the minor $\mathbf{M}_{345}^{234}$ is defined as $\mathbf{A}_{345}^{234}$; i.e., $\mathrm{cof}(\mathbf{M}_{345}^{234}) \stackrel{\triangle}{=} \mathbf{A}_{345}^{234}$, where

$$\mathbf{A}_{345}^{234} = (-1)^{(2+3+4)+(3+4+5)} \overline{\mathbf{M}}_{126\dots n}^{156\dots n} = -\overline{\mathbf{M}}_{126\dots n}^{156\dots n}, \tag{8}$$

since $(-1)^{9+12} = (-1)^{21} = -1$.

In the simple case of a (3×3)-determinant, the co-factor of the element a_n^m is $\mathbf{A}_n^m$; i.e., $\mathrm{cof}(a_n^m) \stackrel{\triangle}{=} \mathbf{A}_n^m$, where

$$\mathbf{A}_n^m = (-1)^{m+n} \overline{\mathbf{M}}_{ij}^{kl}, \tag{9}$$

where $i \neq j \neq n$ and $k \neq l \neq m$. Thus,

$$\begin{aligned} \mathbf{A}_2^3 &= (-1)^{3+2}\overline{\mathbf{M}}_{13}^{12} , \\ &= - \begin{vmatrix} a_1^1 & a_1^2 \\ a_3^1 & a_3^2 \end{vmatrix} , \end{aligned} \tag{10}$$

and

$$\begin{aligned} \mathbf{A}_3^2 &= (-1)^{2+3}\overline{\mathbf{M}}_{12}^{13}, \\ &= - \begin{vmatrix} a_1^1 & a_1^3 \\ a_2^1 & a_2^3 \end{vmatrix} . \end{aligned} \tag{11}$$

Consider now a (4×4)-determinant

$$|\mathbf{A}| = \begin{vmatrix} a_1^1 & a_1^2 & a_1^3 & a_1^4 \\ a_2^1 & a_2^2 & a_2^3 & a_2^4 \\ a_3^1 & a_3^2 & a_3^3 & a_3^4 \\ a_4^1 & a_4^2 & a_4^3 & a_4^4 \end{vmatrix} . \tag{12}$$

In this example the minor whose elements common to the 1-st and 2-nd columns and the 2-nd and 3-rd rows are

$$\mathbf{M}_{23}^{12} = \begin{vmatrix} a_2^1 & a_2^2 \\ a_3^1 & a_3^2 \end{vmatrix} . \tag{13}$$

The cofactor of this minor is

$$\begin{aligned} \mathbf{A}_{23}^{12} &= (-1)^{(1+2)+(2+3)}\bar{\mathbf{M}}_{14}^{34} , \\ &= \begin{vmatrix} a_1^3 & a_1^4 \\ a_4^3 & a_4^4 \end{vmatrix} , \end{aligned} \tag{14}$$

since $(-1)^{3+5} = (-1)^8 = 1$.

As another example, let the minor of the (4×4)-determinant be

$$\mathbf{M}_{23}^{34} = \begin{vmatrix} a_2^3 & a_2^4 \\ a_3^3 & a_3^4 \end{vmatrix} . \tag{15}$$

The co-factor of this minor is

$$\begin{aligned} \mathbf{A}_{23}^{34} &= (-1)^{(3+4)+(2+3)}\bar{\mathbf{M}}_{14}^{12} , \\ &= \begin{vmatrix} a_1^1 & a_1^2 \\ a_4^1 & a_4^2 \end{vmatrix} , \end{aligned} \tag{16}$$

because $(-1)^{7+5} = (-1)^{12} = 1$.

We now give an inductive definition for the evaluation of a determinant, even though it is not that useful for practical purposes .

Definition 1. Consider an $(n \times n)$-matrix with elements a_n^m and corresponding co-factors $\mathbf{A}_n^m$. Then, in terms of the elements of the first row

$$\begin{aligned} |\mathbf{A}| &= a_1^1\mathbf{A}_1^1 + a_1^2\mathbf{A}_1^2 + a_1^3\mathbf{A}_1^3 + \ldots + a_1^n\mathbf{A}_1^n, \\ &= \sum_{p=1}^{n} a_1^p\mathbf{A}_1^p. \end{aligned} \tag{17}$$

In terms of the elements of the second row, the determinant is

$$
\begin{aligned}
|\mathbf{A}| &= a_2^1\mathbf{A}_2^1 + a_2^2\mathbf{A}_2^2 + a_2^3\mathbf{A}_2^3 + \ldots + a_2^n\mathbf{A}_2^n, \\
&= \sum_{p=1}^{n} a_2^p\mathbf{A}_2^p. \qquad (18)
\end{aligned}
$$

In general, the value of the determinant in terms of the elements of the k-th row is

$$
\begin{aligned}
|\mathbf{A}| &= a_k^1\mathbf{A}_k^1 + a_k^2\mathbf{A}_k^2 + a_k^3\mathbf{A}_k^3 + \ldots + a_k^n\mathbf{A}_k^n\ , \\
&= \sum_{p=1}^{n} a_k^p\mathbf{A}_k^p\ , \qquad 1 \le k \le n. \qquad (19)
\end{aligned}
$$

This expansion formula states that the value of an $(n \times n)$-determinant depends on n determinants each of $(n-1)$ order. Each of these $(n-1)$-order determinants depend on $(n-1)$ determinants of order $(n-2)$ each, and so on. If this process is continued then we finally end up with a determinant of order 1. For this reason it is called an *inductive* definition and is not very practical for determinants of order $n \ge 4$.

Consider a (2×2) matrix

$$
[\mathbf{A}] = \begin{bmatrix} a_1^1 & a_1^2 \\ a_2^1 & a_2^2 \end{bmatrix}. \qquad (20)
$$

The determinant of $[a_i^j]$ in terms of the elements of the first row is

$$
|\mathbf{A}| = a_1^1\mathbf{A}_1^1 + a_1^2\mathbf{A}_1^2, \qquad (21)
$$

where

$$
\mathbf{A}_1^1 = (-1)^{1+1}\overline{\mathbf{M}}_2^2 = a_2^2, \qquad (22)
$$

$$
\mathbf{A}_1^2 = (-1)^{2+1}\overline{\mathbf{M}}_2^1 = -a_2^1, \qquad (23)
$$

and therefore

$$
|\mathbf{A}| = a_1^1a_2^2 - a_1^2a_2^1\ . \qquad (24)
$$

Writing the expansion in terms of the elements of the second row, we find

$$
\begin{aligned}
|\mathbf{A}| &= a_2^1\mathbf{A}_2^1 + a_2^2\mathbf{A}_2^2\ , \\
&= (-1)^{1+2}a_2^1\overline{\mathbf{M}}_1^2 + (-1)^{2+2}a_2^2\overline{\mathbf{M}}_1^1\ , \\
&= -a_2^1a_1^2 + a_2^2a_1^1 = a_1^1a_2^2 - a_1^2a_2^1\ . \qquad (25)
\end{aligned}
$$

We can also carry out the expansion in terms of the elements of first column or the second column. Thus, using the first column, we find

$$
\begin{aligned}
|\mathbf{A}| &= a_1^1\mathbf{A}_1^1 + a_2^1\mathbf{A}_2^1\ , \\
&= (-1)^{1+1}a_1^1\overline{\mathbf{M}}_2^2 + (-1)^{1+2}a_2^1\overline{\mathbf{M}}_1^2\ , \\
&= a_1^1a_2^2 - a_1^2a_2^1\ , \qquad (26)
\end{aligned}
$$

which is the same value as the one obtained before. In terms of the elements of the second column, the value of the determinant is

$$
\begin{aligned}
|\mathbf{A}| &= a_1^2\mathbf{A}_1^2 + a_2^2\mathbf{A}_2^2\ , \\
&= (-1)^{2+1}a_1^2\overline{\mathbf{M}}_2^1 + (-1)^{2+2}a_2^2\overline{\mathbf{M}}_1^1\ , \\
&= -a_1^2a_2^1 + a_1^1a_2^2, \qquad (27)
\end{aligned}
$$

which again is the same value as the one obtained earlier.

Note that if we take elements of one row and multiply each element in this row with the corresponding co-factors of the elements located in a different row and take the sum of the products, the result adds up to zero. As a simple example, let the elements in the first row be a_1^1, a_1^2. The successive cofactors of the elements in the second row are $\mathbf{A}_2^1 = -\overline{\mathbf{M}}_1^2 = -a_1^2$, and $\mathbf{A}_2^2 = -\overline{\mathbf{M}}_1^1 = -a_1^1$. Therefore, taking their product and then adding the two we find

$$a_1^1\mathbf{A}_2^1 + a_1^2\mathbf{A}_2^2 = -a_1^1 a_1^2 + a_1^2 a_1^1 \equiv 0. \tag{28}$$

Thus, if we multiply the elements of one row with the co-factors of another row, then the sum adds up to zero. Therefore, to find the non-trivial value of the determinant, the elements and the associated co-factors should belong to the same row or same column.

We now consider a (3×3) determinant

$$|\mathbf{A}| = \begin{vmatrix} a_1^1 & a_1^2 & a_1^3 \\ a_2^1 & a_2^2 & a_2^3 \\ a_3^1 & a_3^2 & a_3^3 \end{vmatrix}. \tag{29}$$

Using expansion in terms of the elements of the first (or second, or third) row, we find that the value of the determinant is:

$$\begin{aligned} |\mathbf{A}| &= a_1^1\mathbf{A}_1^1 + a_1^2\mathbf{A}_1^2 + a_1^3\mathbf{A}_1^3, \\ &= (-1)^{1+1}a_1^1\bar{\mathbf{M}}_{23}^{23} + (-1)^{2+1}a_1^2\bar{\mathbf{M}}_{23}^{13} + (-1)^{3+1}a_1^3\bar{\mathbf{M}}_{23}^{12}, \\ &= a_1^1\begin{vmatrix} a_2^2 & a_2^3 \\ a_3^2 & a_3^3 \end{vmatrix} - a_1^2\begin{vmatrix} a_2^1 & a_2^3 \\ a_3^1 & a_3^3 \end{vmatrix} + a_1^3\begin{vmatrix} a_2^1 & a_2^2 \\ a_3^1 & a_3^2 \end{vmatrix}, \\ &= (a_1^1a_2^2a_3^3 + a_1^2a_2^3a_3^1 + a_1^3a_2^1a_3^2) - (a_1^1a_2^3a_3^2 + a_1^2a_2^1a_3^3 + a_1^3a_2^2a_3^1). \end{aligned} \tag{30}$$

The same value is obtained when we use expansion in terms of the elements of the second or third row.

Remark 1. Note the subscripts which represent the rows. The subscripts in each element of the product are in the natural order $1, 2, 3$. The column indices are $(1, 2, 3)$; $(2, 3, 1)$; $(3, 1, 2)$; and $(1, 3, 2)$; $(2, 1, 3)$; $(3, 2, 1)$. We note that $(2, 3, 1)$ and $(3, 1, 2)$ are even permutations of $(1, 2, 3)$; and $(1, 3, 2)$; $(2, 1, 3)$; $(3, 2, 1)$ are odd permutations of $(1, 2, 3)$. We further note that for even permutations the multiplicative constant is $(+1)$; in the case of odd permutations the multiplicative constant is (-1). The general rule for the signature of a permutation is given in terms of inversions, which we will discuss at an appropriate place in this discourse.

Remark 2. We now realign the product terms in such a manner that the superscripts representing the columns are aligned in the natural order. Thus, after rearrangement, we find

$$|\mathbf{A}| = (a_1^1a_2^2a_3^3 + a_3^1a_1^2a_2^3 + a_2^1a_3^2a_1^3) - (a_1^1a_3^2a_2^3 + a_2^1a_1^2a_3^3 + a_3^1a_2^2a_1^3). \tag{31}$$

In this case the row subscripts $(1, 2, 3)$; $(3, 1, 2)$; $(2, 3, 1)$ are even permutations of $(1, 2, 3)$ and the row subscripts $(1, 3, 2)$; $(2, 1, 3)$; $(3, 2, 1)$ are odd permutations of $(1, 2, 3)$. Therefore in the first grouping the multiplicative constant is $(+1)$ and in the second grouping the multiplicative constant is (-1).

The important fact is that this rearrangement reflects expansion of the (3×3)-determinant in terms of the elements of the j-th column, where the index j may be $1, 2$ or 3. Thus, for $j = 1$,

$$\begin{aligned} |\mathbf{A}| &= a_1^1\begin{vmatrix} a_2^2 & a_3^2 \\ a_2^3 & a_3^3 \end{vmatrix} - a_2^1\begin{vmatrix} a_1^2 & a_1^3 \\ a_3^2 & a_3^3 \end{vmatrix} + a_3^1\begin{vmatrix} a_1^2 & a_1^3 \\ a_2^2 & a_2^3 \end{vmatrix}, \\ &= a_1^1\mathbf{A}_1^1 + a_2^1\mathbf{A}_2^1 + a_3^1\mathbf{A}_3^1, \end{aligned} \tag{32}$$

In general, we can show that for a (3×3)-determinant there are 3! ways to find the non-trivial value of the determinant. Any other set of 3! arrangements will lead to the trivial value zero. We first list the 3! ways which lead to the non-trivial result. These are:

1. Expansion in terms of the elements of the rows:

$$\begin{aligned} |\mathbf{A}| &= a_1^1\mathbf{A}_1^1 + a_1^2\mathbf{A}_1^2 + a_1^3\mathbf{A}_1^3 \ , \ \text{(1-st row)} \\ &= a_2^1\mathbf{A}_2^1 + a_2^2\mathbf{A}_2^2 + a_2^3\mathbf{A}_2^3 \ , \ \text{(2-nd row)} \\ &= a_3^1\mathbf{A}_3^1 + a_3^2\mathbf{A}_3^2 + a_3^3\mathbf{A}_3^3 \ . \ \text{(3-rd row)} \end{aligned} \tag{33}$$

2. Expansion in terms of the elements of the columns:

$$\begin{aligned} |\mathbf{A}| &= a_1^1\mathbf{A}_1^1 + a_2^1\mathbf{A}_2^1 + a_3^1\mathbf{A}_3^1 \ , \ \text{(1-st column)} \\ &= a_1^2\mathbf{A}_1^2 + a_2^2\mathbf{A}_2^2 + a_3^2\mathbf{A}_3^2 \ , \ \text{(2-nd column)} \\ &= a_1^3\mathbf{A}_1^3 + a_2^3\mathbf{A}_2^3 + a_3^3\mathbf{A}_3^3 \ . \ \text{(3-rd column)} \end{aligned} \tag{34}$$

There are another 3! arrangements which yield the trivial value zero. These are

$$\begin{aligned} a_1^1\mathbf{A}_2^1 + a_1^2\mathbf{A}_2^2 + a_1^3\mathbf{A}_2^3 &\equiv 0 \ , \\ a_1^1\mathbf{A}_3^1 + a_1^2\mathbf{A}_3^2 + a_1^3\mathbf{A}_3^3 &\equiv 0 \ , \\ a_2^1\mathbf{A}_1^1 + a_2^2\mathbf{A}_1^2 + a_2^3\mathbf{A}_1^3 &\equiv 0 \ , \\ a_2^1\mathbf{A}_3^1 + a_2^2\mathbf{A}_3^2 + a_2^3\mathbf{A}_3^3 &\equiv 0 \ , \\ a_3^1\mathbf{A}_1^1 + a_3^2\mathbf{A}_1^2 + a_3^3\mathbf{A}_1^3 &\equiv 0 \ , \\ a_3^1\mathbf{A}_2^1 + a_3^2\mathbf{A}_2^2 + a_3^3\mathbf{A}_2^3 &\equiv 0 \ . \end{aligned} \tag{35}$$

We note that the expansion of a (2×2)-determinant involves 2! terms, and the expansion of a (3×3)-determinant involves $3 \times 2! = 3!$ terms, because the expansion invlolves three (2×2)-minors, where each (2×2) minor has two terms. Using inductive reasoning we find that a (4×4)-determinant will have $4 \times 3! = 4!$ terms, because the expansion will involve four (3×3)-minors. Thus we find:

$(n \times n)$-determinant	Number of terms
(2×2)	$2! = 2,$
(3×3)	$3! = 6,$
(4×4)	$4! = 24,$
(5×5)	$5! = 120,$
(6×6)	$6! = 720,$
(7×7)	$7! = 5,040,$
(8×8)	$8! = 40,320,$
(9×9)	$9! = 362,880,$
(10×10)	$10! = 3,628,800.$

Obviously the number of terms in an higher order determinant increases as the order n increases. It is well known that the growth of the factorial function $n!$ is much greater compared to the growth of an exponential function e^n. In fact

$$\lim_{n \to \infty} \frac{\mathrm{e}^n}{n!} \to 0, \tag{36}$$

which verifies our assertion that $n!$ for large n involves a very large number of terms. In such cases when $n > 3$, the inductive algorithm based on **Definition 1** is not very practical. This process requires some modification to make it simpler. We describe one such procedure now.

2 GENERALIZED LAPLACE EXPANSION FOR $n > 3$

So far we showed how to carry out the expansion of a determinant in terms of the elements of a row (column). We now show that in the case of higher order determinants of order $n > 3$, we can easily carry out the expansion of the determinant in terms of all possible row (column) (2×2)-minors or possibly (3×3)- or (4×4)-minors.

As an example we first consider a (4×4)-determinant:

$$|\mathbf{A}| = \begin{vmatrix} a_1^1 & a_1^2 & a_1^3 & a_1^4 \\ a_2^1 & a_2^2 & a_2^3 & a_2^4 \\ a_3^1 & a_3^2 & a_3^3 & a_3^4 \\ a_4^1 & a_4^2 & a_4^3 & a_4^4 \end{vmatrix}. \tag{37}$$

Take now any two rows (columns) and for convenience let these be the 1-st and 2-nd rows. In a (4×4)-determinant there exist $\binom{4}{2} = \frac{4!}{(4-2)!2!} = 3!$ (2×2)-minors, whose elements lie in the 1-st and 2-nd rows. These (2×2)-minors are

$$\begin{array}{ccc} \mathbf{M}_{12}^{12}, & \mathbf{M}_{12}^{13}, & \mathbf{M}_{12}^{14}, \\ & \mathbf{M}_{12}^{23}, & \mathbf{M}_{12}^{24}, \\ & & \mathbf{M}_{12}^{34}, \end{array}$$

where the dead subscripts $(1,2)$ represent elements in the first and second rows and the live subscripts in the ascending order $(1,2)$, $(1,3)$, $(1,4)$; $(2,3)$, $(2,4)$; $(3,4)$ represent the column indices. Corresponding to these minors we have 6-(2×2)-co-factors

$$\begin{array}{ccc} \mathbf{A}_{12}^{12}, & \mathbf{A}_{12}^{13}, & \mathbf{A}_{12}^{14}, \\ & \mathbf{A}_{12}^{23}, & \mathbf{A}_{12}^{24}, \\ & & \mathbf{A}_{12}^{34}. \end{array}$$

These cofactors are related to the complementary minors as described below: [1]

[1] If in these formula we read $\mathbf{A}_{pq}^{mn}$ as the cofactor of the minor $\mathbf{M}_{mn}^{pq}$ with the role of rows and columns interchanged and if we use Einstein's summation convention and generalized Kronecker delta, then these results can be concisely stated as:

1. Co-factor of a (2×2)-minor in a (4×4)-determinant,

$$\mathrm{cof}(\mathbf{M}_{mn}^{pq}) \triangleq \mathbf{A}_{pq}^{mn} = \frac{1}{2!}\, \delta_{pq\,i\,j}^{mnrs} a_r^i a_s^j.$$

2. Co-factor of a (2×2)-minor in a (5×5)-determinant,

$$\mathrm{cof}(\mathbf{M}_{mn}^{pq}) \triangleq \mathbf{A}_{pq}^{mn} = \frac{1}{3!}\, \delta_{pq\,i\,j\,k}^{mnrst} a_r^i a_s^j a_t^k.$$

In each case the determinant is

$$\mathbf{M}_{rs}^{mn} \mathbf{A}_{mn}^{pq} = \mathbf{M}_{mn}^{pq} \mathbf{A}_{rs}^{mn} = \mathbf{A} \delta_{rs}^{pq}.$$

We may note that the generalized Kronecker delta has k superscripts and k subscripts. These superscripts and subscripts run from 1 to n and each set is alternating. If the superscripts are distinct from each other and the subscripts are the same set of numbers as the superscripts, then the value of the generalized Kronecker delta is either $+1$ or -1 depending upon whether an even or an odd permutation is required to rearrange the superscripts in the same order as the subscripts. In all other cases the value of the generalized Kronecker delta is zero.

$$\mathbf{A}_{12}^{12} = (-1)^{(1+2)+(1+2)}\overline{\mathbf{M}}_{34}^{34} = +\begin{vmatrix} a_3^3 & a_3^4 \\ a_4^3 & a_4^4 \end{vmatrix}, \tag{38}$$

$$\mathbf{A}_{12}^{13} = (-1)^{(1+2)+(1+3)}\overline{\mathbf{M}}_{34}^{24} = -\begin{vmatrix} a_3^2 & a_3^4 \\ a_4^2 & a_4^4 \end{vmatrix}, \tag{39}$$

$$\mathbf{A}_{12}^{14} = (-1)^{(1+2)+(1+4)}\overline{\mathbf{M}}_{34}^{23} = +\begin{vmatrix} a_3^2 & a_3^3 \\ a_4^2 & a_4^3 \end{vmatrix}, \tag{40}$$

$$\mathbf{A}_{12}^{23} = (-1)^{(1+2)+(2+3)}\overline{\mathbf{M}}_{34}^{14} = +\begin{vmatrix} a_3^1 & a_3^4 \\ a_4^1 & a_4^4 \end{vmatrix}, \tag{41}$$

$$\mathbf{A}_{12}^{24} = (-1)^{(1+2)+(2+4)}\overline{\mathbf{M}}_{34}^{13} = -\begin{vmatrix} a_3^1 & a_3^3 \\ a_4^1 & a_4^3 \end{vmatrix}, \tag{42}$$

$$\mathbf{A}_{12}^{34} = (-1)^{(1+2)+(3+4)}\overline{\mathbf{M}}_{34}^{12} = +\begin{vmatrix} a_3^1 & a_3^2 \\ a_4^1 & a_4^2 \end{vmatrix}. \tag{43}$$

Hence the expansion of the (4×4)-determinant in terms of all possible distinct (2×2)-minors of the first and second rows gives us the value of the determinant

$$\begin{aligned}
|\mathbf{A}| = \; & \mathbf{M}_{12}^{12}\mathbf{A}_{12}^{12} + \mathbf{M}_{12}^{13}\mathbf{A}_{12}^{13} + \mathbf{M}_{12}^{14}\mathbf{A}_{12}^{14} \\
& + \mathbf{M}_{12}^{23}\mathbf{A}_{12}^{23} + \mathbf{M}_{12}^{24}\mathbf{A}_{12}^{24} \\
& + \mathbf{M}_{12}^{34}\mathbf{A}_{12}^{34},
\end{aligned}$$

$$= \begin{vmatrix} a_1^1 & a_1^2 \\ a_2^1 & a_2^2 \end{vmatrix}\begin{vmatrix} a_3^3 & a_3^4 \\ a_4^3 & a_4^4 \end{vmatrix} - \begin{vmatrix} a_1^1 & a_1^3 \\ a_2^1 & a_2^3 \end{vmatrix}\begin{vmatrix} a_3^2 & a_3^4 \\ a_4^2 & a_4^4 \end{vmatrix}$$

$$+ \begin{vmatrix} a_1^1 & a_1^4 \\ a_2^1 & a_2^4 \end{vmatrix}\begin{vmatrix} a_3^2 & a_3^3 \\ a_4^2 & a_4^3 \end{vmatrix} + \begin{vmatrix} a_1^2 & a_1^3 \\ a_2^2 & a_2^3 \end{vmatrix}\begin{vmatrix} a_3^1 & a_3^4 \\ a_4^1 & a_4^4 \end{vmatrix}$$

$$- \begin{vmatrix} a_1^2 & a_1^4 \\ a_2^2 & a_2^4 \end{vmatrix}\begin{vmatrix} a_3^1 & a_3^3 \\ a_4^1 & a_4^3 \end{vmatrix} + \begin{vmatrix} a_1^3 & a_1^4 \\ a_2^3 & a_2^4 \end{vmatrix}\begin{vmatrix} a_3^1 & a_3^2 \\ a_4^1 & a_4^2 \end{vmatrix},$$

$$= (a_1^1 a_2^2 - a_1^2 a_2^1)(a_3^3 a_4^4 - a_3^4 a_4^3) - (a_1^1 a_2^3 - a_1^3 a_2^1)(a_3^2 a_4^4 - a_3^4 a_4^2)$$

$$+ (a_1^1 a_2^4 - a_1^4 a_2^1)(a_3^2 a_4^3 - a_3^3 a_4^2) + (a_1^2 a_2^3 - a_1^3 a_2^2)(a_3^1 a_4^4 - a_3^4 a_4^1)$$

$$-(a_1^2 a_2^4 - a_1^4 a_2^2)(a_3^1 a_4^3 - a_3^3 a_4^1) + (a_1^3 a_2^4 - a_1^4 a_2^3)(a_3^1 a_4^2 - a_3^2 a_4^1). \tag{44}$$

Hence, expansion of a (4×4)-determinant has been reduced to the problem of 6 products of $(2 \times 2) \times (2 \times 2)$-determinants instead of four (3×3)-determinants. This method of expansion is due to Laplace.

Example 1. Given a (4×4)-determinant

$$|\mathbf{A}| = \begin{vmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \\ 3 & 1 & 2 & 4 \\ 4 & 2 & 1 & 3 \end{vmatrix}, \tag{45}$$

find the value of $|\mathbf{A}|$ using (2×2)-minor blocks.

Using $\binom{4}{2}= 3!$ (2×2)-minors of the first and second rows, we find

$$\begin{aligned}|\mathbf{A}| = &\begin{vmatrix} 1 & 2 \\ 4 & 3 \end{vmatrix}\begin{vmatrix} 2 & 4 \\ 1 & 3 \end{vmatrix} - \begin{vmatrix} 1 & 3 \\ 4 & 2 \end{vmatrix}\begin{vmatrix} 1 & 4 \\ 2 & 3 \end{vmatrix} + \begin{vmatrix} 1 & 4 \\ 4 & 1 \end{vmatrix}\begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} + \begin{vmatrix} 2 & 3 \\ 3 & 2 \end{vmatrix}\begin{vmatrix} 3 & 4 \\ 4 & 3 \end{vmatrix} \\ &- \begin{vmatrix} 2 & 4 \\ 3 & 1 \end{vmatrix}\begin{vmatrix} 3 & 2 \\ 4 & 1 \end{vmatrix} + \begin{vmatrix} 3 & 4 \\ 2 & 1 \end{vmatrix}\begin{vmatrix} 3 & 1 \\ 4 & 2 \end{vmatrix}, \\ |\mathbf{A}| = &(3-8)(6-4) - (2-12)(3-8) + (1-16)(1-4) + (4-9)(9-16) \\ &- (2-12)(3-8) + (3-8)(6-4), \\ &= -5\times 2 - 10\times 5 + 15\times 3 + 5\times 7 - 10\times 5 - 5\times 2, \\ &= -(10+50+50+10) + (45+35), \\ &= -120+70;\end{aligned} \tag{46}$$

$$|\mathbf{A}| = -50. \tag{47}$$

In the next example we find the value of a (5×5)-determinant by using (2×2)-minors. The total number of (2×2)-minors in this case is $\binom{5}{2}= \frac{5!}{3!2!} = 10$. Note that if instead of (2×2)-minors we use (3×3)-minors, then also the total number of (3×3)-minors is 10, because Bernoulli coefficients have the symmetry property $\binom{5}{3}=\binom{5}{2}= 10$, and thus no advantage is gained. In the case of (2×2)-minors the various combinations are

$$\{12, 13, 14, 15; 23, 24, 25; 34, 35; 45\}.$$

When using (3×3)-minors the various distinct combinations are

$$\{123, 124, 125, 134, 135, 145; 234, 235, 245; 345\}.$$

However, expansion of the (5×5)-determinant in terms of (2×2)-minors is equivalent to expansion in terms of the (3×3)-minors, because one is the complement of the other. Hence, it suffices to carry out expansion in terms of (2×2)-minors.

Example 2. Given a (5×5)-determinant

$$|\mathbf{A}| = \begin{vmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 4 & 3 & 2 & 1 \\ 4 & 3 & 2 & 1 & 5 \\ 2 & 1 & 4 & 3 & 5 \\ 3 & 5 & 2 & 1 & 4 \end{vmatrix}, \tag{48}$$

find the value of $|\mathbf{A}|$ using (2×2)-minor blocks.

Using (2×2)-minors, the expansion is

$$
\begin{aligned}
|\mathbf{A}| = &\left[\begin{vmatrix} 1 & 2 \\ 5 & 4 \end{vmatrix}\begin{vmatrix} 2 & 1 & 5 \\ 4 & 3 & 5 \\ 2 & 1 & 4 \end{vmatrix} - \begin{vmatrix} 1 & 3 \\ 5 & 3 \end{vmatrix}\begin{vmatrix} 3 & 1 & 5 \\ 1 & 3 & 5 \\ 5 & 1 & 4 \end{vmatrix} + \begin{vmatrix} 1 & 4 \\ 5 & 2 \end{vmatrix}\begin{vmatrix} 3 & 2 & 5 \\ 1 & 4 & 5 \\ 5 & 2 & 4 \end{vmatrix}\right. \\
&\left. - \begin{vmatrix} 1 & 5 \\ 5 & 1 \end{vmatrix}\begin{vmatrix} 3 & 2 & 1 \\ 1 & 4 & 3 \\ 5 & 2 & 1 \end{vmatrix}\right] \\
&+ \left[\begin{vmatrix} 2 & 3 \\ 4 & 3 \end{vmatrix}\begin{vmatrix} 4 & 1 & 5 \\ 2 & 3 & 5 \\ 3 & 1 & 4 \end{vmatrix} - \begin{vmatrix} 2 & 4 \\ 4 & 2 \end{vmatrix}\begin{vmatrix} 4 & 2 & 5 \\ 2 & 4 & 5 \\ 3 & 2 & 4 \end{vmatrix} + \begin{vmatrix} 2 & 5 \\ 4 & 1 \end{vmatrix}\begin{vmatrix} 4 & 2 & 1 \\ 2 & 4 & 3 \\ 3 & 2 & 1 \end{vmatrix}\right] \\
&+ \left[\begin{vmatrix} 3 & 4 \\ 3 & 2 \end{vmatrix}\begin{vmatrix} 4 & 3 & 5 \\ 2 & 1 & 5 \\ 3 & 5 & 4 \end{vmatrix} - \begin{vmatrix} 3 & 5 \\ 3 & 1 \end{vmatrix}\begin{vmatrix} 4 & 3 & 1 \\ 2 & 1 & 3 \\ 3 & 5 & 1 \end{vmatrix}\right] \\
&+ \begin{vmatrix} 4 & 5 \\ 2 & 1 \end{vmatrix}\begin{vmatrix} 4 & 3 & 2 \\ 2 & 1 & 4 \\ 3 & 5 & 2 \end{vmatrix}.
\end{aligned}
$$

Evaluating the (2×2)- and (3×3)-determinants which enter in the expansion above,we find

$$
\begin{aligned}
|\mathbf{A}| &= (-6) \times (-2) - (-12) \times (-28) + (-18) \times (-30) - (-24) \times 4 \\
&\quad + (-6) \times 0 - (-12) \times (-2) + (-18) \times (-2) + (-6) \times (-8) \\
&\quad - (-12) \times (-28) + (-6) \times (-34), \\
&= 12 - 336 + 540 + 96 + 0 - 24 + 36 + 48 - 336 + 204; \\
|\mathbf{A}| &= 240. \qquad (49)
\end{aligned}
$$

Exercise 1 : Evaluate the following determinant first by using the (2×2)-minors of 3-rd and 4-th row and then reevaluate it by using (2×2)-minors of 3-rd and 4-th column

$$
|\mathbf{A}| = \begin{vmatrix} 6 & 4 & -6 & 2 \\ 7 & 1 & 2 & 4 \\ 8 & -3 & 5 & -7 \\ 6 & 5 & 4 & 3 \end{vmatrix}. \qquad (50)
$$

We now list without proof some of the important properties of determinants. Examples of these properties are provided at the end of Exercises (see Section 5).

Property 1. Let a square matrix $[\mathbf{A}]^T$ be the transpose of $[\mathbf{A}]$. Then

$$
|\mathbf{A}^T| = |\mathbf{A}|. \qquad (51)
$$

Property 2. For a triangular matrix of order n, the determinant is the product of its diagonal elements.

Property 3. If each element of a row (column) is multiplied by a constant C, the value of the determinant is multiplied by this constant.

Property 4. Interchanging any two adjacent rows or columns changes the sign of the determinant.

Property 5. If any two rows or any two columns are identically equal, then the value of the determinant is zero.

Property 6. If the determinant of a square matrix is zero, then the row vectors or column vectors are not linearly independent.

Property 7. The product of the determinants of two square matrices is equal to the determinant of the product of the two matrices; i.e., if **A** and **B** are two square matrices of the same order, then

$$|\mathbf{A}| \times |\mathbf{B}| = |\mathbf{A} \times \mathbf{B}|. \tag{52}$$

Exercise : Verify the equality:

$$\begin{vmatrix} 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{vmatrix} \times \begin{vmatrix} 3 & 4 & 5 \\ 1 & 2 & 1 \\ 5 & 3 & 4 \end{vmatrix} = \left| \begin{bmatrix} 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{bmatrix} \times \begin{bmatrix} 3 & 4 & 5 \\ 1 & 2 & 1 \\ 5 & 3 & 4 \end{bmatrix} \right|. \tag{53}$$

Determinants have many other useful properties. One can easily find these seven properties cited above and many more in any good book in Linear Algebra[2].

3 APPLICATIONS

In this section we give a few simple applications of determinants.

1. Let us assume that we are given three vectors **A**, **B** and **C** in three dimensional Cartesian space. The absolute value of the volume subtended by these three vectors is

$$|\mathbf{V}| = \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}. \tag{54}$$

It is easy to verify that this is exactly the scalar product of vector **A** with the vector product $\mathbf{B} \times \mathbf{C}$; i.e.,

$$|\mathbf{V}| = \mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}). \tag{55}$$

If the determinant is zero it means that the volume enclosed by the three vectors is zero. That implies that the three vectors lie in the same plane; i.e., they are co-planar. In other words these three vectors are not linearly independent; i.e., they are dependent.

We also note that in the case of dot-cross product

$$\begin{aligned} &\mathbf{A}.(\mathbf{B} \times \mathbf{C}) = \mathbf{B}.(\mathbf{C} \times \mathbf{A}) = \mathbf{C}.(\mathbf{A} \times \mathbf{B}) \\ &= -\mathbf{A}.(\mathbf{C} \times \mathbf{B}) = -\mathbf{B}.(\mathbf{A} \times \mathbf{C}) = -\mathbf{C}.(\mathbf{B} \times \mathbf{A}). \end{aligned} \tag{56}$$

Similar transpositions can also be carried out in the case of determinants. Interchanging any two rows or any two columns leads to a change of sign of the determinant. Thus,

$$\begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix} = (-) \begin{vmatrix} B_x & B_y & B_z \\ A_x & A_y & A_z \\ C_x & C_y & C_z \end{vmatrix} = (-)^2 \begin{vmatrix} B_x & B_y & B_z \\ C_x & C_y & C_z \\ A_x & A_y & A_z \end{vmatrix}, \text{ etc.} \tag{57}$$

[2]Hoffman, K. and R.Kunze, *Linear Algebra*, Prentice-Hall, Inc., Englewood Cliffs, NJ (1961)

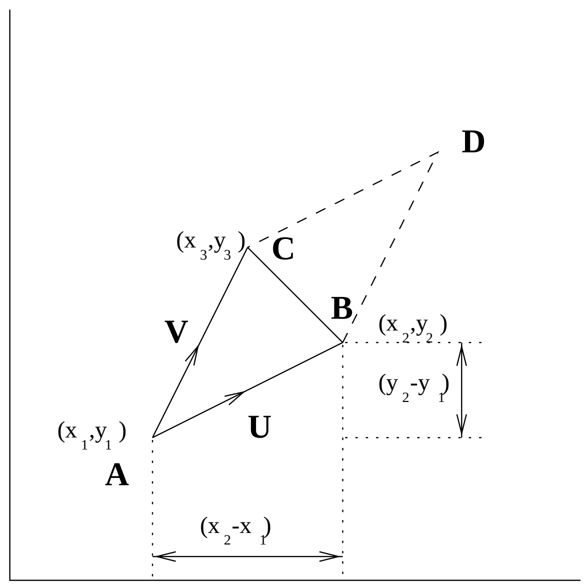
2. Consider two vectors **A** and **B** in the xy-plane. Their components are (A_x, A_y) and (B_x, B_y). The absolute value of the area of the parallelogram formed by these two vectors is

$$|\text{Area}| = \begin{vmatrix} A_x & A_y \\ B_x & B_y \end{vmatrix}. \tag{58}$$

3. Area of a triangle:

   Let the coordinates of three points in a plane be

$$\text{A: } (x_1, y_1),\ \text{B: } (x_2, y_2),\ \text{C: } (x_3, y_3)$$



In terms of vectors **U** and **V** we have

$$\mathbf{U} = (x_2 - x_1, y_2 - y_1),\ \mathbf{V} = (x_3 - x_1, y_3 - y_1). \tag{59}$$

The absolute value of the area of the triangle ABC can now be written as

$$|\mathbf{A}| = \frac{1}{2} \begin{vmatrix} (x_2 - x_1) & (y_2 - y_1) \\ (x_3 - x_1) & (y_3 - y_1) \end{vmatrix}, \tag{60}$$

where the multiplicative factor 1/2 is necessary because we are interested in the area of a triangle and not of the parallelogram ABCD. If we expand this (2×2)-determinant, we get

$$\begin{aligned} |\mathbf{A}| &= \frac{1}{2}[(x_2 - x_1)(y_3 - y_1) - (x_3 - x_1)(y_2 - y_1)], \\ &= \frac{1}{2}[x_2y_3 - x_2y_1 - x_1y_3 + x_1y_1 - x_3y_2 + x_1y_2 + x_3y_1 - x_1y_1], \\ &= \frac{1}{2}[(x_2y_3 - x_3y_2) - (x_1y_3 - x_3y_1) + (x_1y_2 - x_2y_1)]; \\ &= \frac{1}{2} \begin{vmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix}. \end{aligned} \tag{61}$$

Thus, the area of a triangle in xy-plane can also be written in terms of a (3×3)-determinant.

If the points (x_1, y_1), (x_2, y_2), (x_3, y_3) lie on the same line then the area of the triangle whose three vertices lie on these points is zero. Therefore

$$\begin{vmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix} = 0. \tag{62}$$

This is the equation of a straight line passing through the three given points.

Using determinants, we can also write the equations of various geometrical objects. As an example the equation of a circle passing through three given points (x_1, y_1), (x_2, y_2), (x_3, y_3) can be written in terms of a (4×4)-determinant. Similarly, in three-dimensional space we can write the equation of a plane passing through three given points (a_1, a_2, a_3), (b_1, b_2, b_3) and (c_1, c_2, c_3) by using a (4×4)-determinant:

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ x & a_1 & a_2 & a_3 \\ y & b_1 & b_2 & b_3 \\ z & c_1 & c_2 & c_3 \end{vmatrix} = 0. \tag{63}$$

4 EXERCISES

1. Evaluate the following (2×2)-determinants:

   (a) $\begin{vmatrix} 6 & 3 \\ 5 & 2 \end{vmatrix}$,

   (b) $\begin{vmatrix} 3 & 2 \\ 8 & 5 \end{vmatrix}$,

   (c) $\begin{vmatrix} n & (n+1) \\ (n-1) & n \end{vmatrix}$,

   (d) $\begin{vmatrix} \cos\theta + i\sin\theta & 1 \\ -1 & \cos\theta - i\sin\theta \end{vmatrix}$, $i = \sqrt{-1}$.

2. Evaluate the following (3×3)-determinants:

   (a) $\begin{vmatrix} 2 & 1 & 3 \\ 1 & 4 & 3 \\ 5 & 3 & 2 \end{vmatrix}$,

   (b) $\begin{vmatrix} 4 & 2 & -1 \\ 5 & 3 & -2 \\ 3 & 2 & 1 \end{vmatrix}$,

   (c) $\begin{vmatrix} 1 & 1 & 1 \\ 5 & 7 & 9 \\ 5^2 & 7^2 & 9^2 \end{vmatrix}$,

   (d) $\begin{vmatrix} 2 & 0 & 3 \\ 7 & 1 & 6 \\ 6 & 0 & 5 \end{vmatrix}$.

3. Evaluate the following (4×4)-determinants, using (2×2)-minors:

   (a) $\begin{vmatrix} 5 & 2 & 1 & 7 \\ 3 & 0 & 0 & 2 \\ 1 & 3 & 4 & 5 \\ 4 & 0 & 3 & 7 \end{vmatrix}$,

   (b) $\begin{vmatrix} 1 & 1 & 3 & 4 \\ 2 & 1 & 1 & 8 \\ 3 & 1 & 1 & 2 \\ 4 & 4 & 7 & 8 \end{vmatrix}$.

4. Evaluate the following (5×5)-determinant, using (2×2)-minors of the

   (a) 1-st and 2-nd row,
   (b) 3-rd and 4-th row,
   (c) 2-nd and 3-rd column,
   (d) 4-th and 5-th column:

$$\mathbf{A} = \begin{vmatrix} -2 & 5 & 0 & -1 & 3 \\ 1 & 0 & 3 & 7 & -2 \\ 3 & -1 & 0 & 5 & -5 \\ 2 & 6 & -4 & 7 & 2 \\ 0 & -3 & -1 & 2 & 3 \end{vmatrix}.$$

5. The numbers 20604, 53227, 25755, 20927 and 78421 are all divisible by 17. Show that the determinant

$$\mathbf{A} = \begin{vmatrix} 2 & 0 & 6 & 0 & 4 \\ 5 & 3 & 2 & 2 & 7 \\ 2 & 5 & 7 & 5 & 5 \\ 2 & 0 & 9 & 2 & 7 \\ 7 & 8 & 4 & 2 & 1 \end{vmatrix},$$

is also divisible by 17.

6. In a determinant of order four, write all the terms which contain the factor a_2^3
   (a) and have a positive sign,
   (b) and have a negative sign.

7. In a (3×3)-determinant, show that if the three columns are linearly dependent, then the three rows are also linearly dependent.

8. Find the co-factor of each element of the given matrix
   (a) $\begin{vmatrix} 2 & 3 \\ -1 & 5 \end{vmatrix}$,
   (b) $\begin{vmatrix} 4 & 0 & 2 \\ 0 & 2 & -2 \\ 1 & 3 & 1 \end{vmatrix}$.

9. Derive Property 2.

10. Derive Property 4.

11. Derive Property 5.

12. Derive Property 7.

13. Find the numbers of inversions in the following permutations:
    (a) $(6, 3, 1, 2, 5, 4)$
    (b) $(1, 9, 6, 3, 2, 5, 4, 7, 8)$
    (c) $(7, 5, 6, 4, 1, 3, 2)$
    (d) $(2, 4, 6, 8, 10, 1, 3, 5, 7, 9)$
    (e) $(1, 3, 5, 7, 9, 2, 4, 6, 8, 10)$

14. Consider the (5×5)-determinant shown in the figure below:

$$\mathbf{A} = \begin{vmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ b_1 & b_2 & b_3 & b_4 & b_5 \\ c_1 & c_2 & c_3 & c_4 & c_5 \\ d_1 & d_2 & d_3 & d_4 & d_5 \\ e_1 & e_2 & e_3 & e_4 & e_5 \end{vmatrix}.$$

Which of the following combinations actually occur in the expansion of the determinant and with what sign?

   (a) $d_3 b_1 c_5 a_2 e_4$,

(b) $b_1c_2d_3a_1e_4$,

(c) $a_2b_3c_1d_5e_4$.

Choose the indices m and n so that in the expansion of the determinant the following terms appear with a positive sign:

(a) $a_2b_mc_nd_5e_1$,

(b) $b_3d_me_na_1c_1$.

15. Show that if a, b and c are real, then the roots of the equation

$$\begin{vmatrix} a-x & b \\ b & c-x \end{vmatrix} = 0,$$

are also real.

16. Evaluate the determinant

(a)

$$\begin{vmatrix} a+ib & c+id \\ -c+id & a-ib \end{vmatrix}, \text{ where } i=\sqrt{-1}\ ;$$

(b)

$$\begin{vmatrix} 1 & 0 & 1+i \\ 0 & 1 & i \\ 1-i & -i & 1 \end{vmatrix};$$

(c)

$$\begin{vmatrix} \cos\alpha & \sin\alpha\cos\beta & \sin\alpha\sin\beta \\ -\sin\alpha & \cos\alpha\cos\beta & \cos\alpha\sin\beta \\ & -\sin\beta & \cos\beta \end{vmatrix}.$$

5 PROPERTIES

Properties of determinants listed elsewhere in these notes are illustrated here with the help of examples. Proof of these and many other properties can be found in books of linear algebra.

Property 1

$$|\mathbf{A}| = \begin{vmatrix} 1 & 3 & 5 \\ 2 & 1 & 4 \\ 1 & 3 & 7 \end{vmatrix} = 1\times(-5) - 3\times 10 + 5\times 5 = -10; \tag{64}$$

$$|\mathbf{A}|^T = \begin{vmatrix} 1 & 2 & 1 \\ 3 & 1 & 3 \\ 5 & 4 & 7 \end{vmatrix} = 1\times(-5) - 2\times 6 + 1\times 7 = -10. \tag{65}$$

Property 2

$$\begin{vmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 3 & 4 & 5 \end{vmatrix} = 1\times 3\times 5. \tag{66}$$

Property 3

$$\begin{vmatrix} 1 & 3 & 5 \\ 4 & 2 & 8 \\ 1 & 3 & 7 \end{vmatrix} = 2 \begin{vmatrix} 1 & 3 & 5 \\ 2 & 1 & 4 \\ 1 & 3 & 7 \end{vmatrix} = 2 \times (-10) = -20. \quad (67)$$

Property 4

$$\begin{vmatrix} 1 & 3 & 5 \\ 2 & 1 & 4 \\ 1 & 3 & 7 \end{vmatrix} = - \begin{vmatrix} 2 & 1 & 4 \\ 1 & 3 & 5 \\ 1 & 3 & 7 \end{vmatrix} = (-1)^2 \begin{vmatrix} 1 & 2 & 4 \\ 3 & 1 & 5 \\ 3 & 1 & 7 \end{vmatrix} = -10. \quad (68)$$

Property 5

$$\begin{vmatrix} 1 & 2 & 4 \\ 2 & 4 & 8 \\ 3 & 5 & 2 \end{vmatrix} = 2 \begin{vmatrix} 1 & 2 & 4 \\ 1 & 2 & 4 \\ 3 & 5 & 2 \end{vmatrix} =$$

$$2[1(4-20) - 2(2-12) + 4(5-6)] = 2[-16+20-4] = 0. \quad (69)$$

Property 6

$$\begin{vmatrix} a & b & c \\ 2a & 2b & 2c \\ r & s & t \end{vmatrix} = 0 \ , \quad (70)$$

because $\{2a, 2b, 2c\} = 2\{a, b, c\}$.

Property 7

$$\begin{aligned} \left| \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} p & q \\ r & s \end{bmatrix} \right| &= \begin{vmatrix} ap+br & aq+bs \\ cp+dr & cq+ds \end{vmatrix} \ , \\ &= (ap+br)(cq+ds) - (aq+bs)(cp+dr) \ , \\ &= apcq + apds + brcq + brds - aqcp - aqdr - bscp - bsdr \ , \\ &= (apds - aqdr) - (bscp - brcq) = ad(ps-qr) - bc(sp-rq) \ , \\ &= (ad-bc)(ps-qr) \ , \\ &= \begin{vmatrix} a & b \\ c & d \end{vmatrix} \times \begin{vmatrix} p & q \\ r & s \end{vmatrix} \ . \end{aligned} \quad (71)$$

It is true in general for any two $(n \times n)$-determinants.

CRAMER'S RULE

For a System of Simultaneous Equations

We have introduced the concept of a determinant and we have also explained some of its basic properties with the help of examples. We next consider the application of these concepts to the solution of a system of linear equations.

We first make the following assumptions:

1. The system of linear equations are non-homogeneous, i.e. they are of the form

$$\mathbf{a}\cdot\mathbf{x}=\mathbf{b},$$

where the vector $\mathbf{b}$ on the right-hand side of the equation is a non-trivial vector such that the scalar product $\mathbf{b}\cdot\mathbf{b}\neq\mathbf{0}$.

2. The system of linear equations form a consistent set so that a non-trivial solution $\mathbf{x}$ exists and is unique. This implies that we assume that the linear equations are linearly independent, which requires that the determinant of the dyadic coefficients $|a^n_m|\neq 0$.

To find the solution of a system of linear equations we make use of an important property of co-factors which we have listed earlier. According to this property, if A^m_i is the co-factor of the dyadic element a^m_i, then using column (or row) expansions, we have

1. $\displaystyle\sum_{m=1}^{n} a^m_i A^m_j = A\delta_{ij},$

2. $\displaystyle\sum_{m=1}^{n} a^i_m A^j_m = A\delta^{ij},$

where $\delta^{ij}(\delta_{ij})$ are *Kronecker deltas* defined as

$$\begin{aligned}\delta^{ij} &= 1 \;\; for\ i=j,\\ &= 0 \;\; for\ i\neq j,\end{aligned}$$

$$\begin{aligned}\delta_{ij} &= 1 \;\; for\ i=j,\\ &= 0 \;\; for\ i\neq j.\end{aligned}$$

We thus see that when the subscript $ij=11,22,33,\cdots,nn$

$$\begin{array}{ccccccccccccc} a^1_1A^1_1 &+& a^2_1A^2_1 &+& a^3_1A^3_1 &+& \cdots &+& a^n_1A^n_1 &=& A\ ,\\ a^1_2A^1_2 &+& a^2_2A^2_2 &+& a^3_2A^3_2 &+& \cdots &+& a^n_2A^n_2 &=& A\ ,\\ \vdots &&&&&&&&&& \vdots\\ a^1_nA^1_n &+& a^2_nA^2_n &+& a^3_nA^3_n &+& \cdots &+& a^n_nA^n_n &=& A\ ,\end{array}$$

and when $ij=12,13,\cdots,1n;21,23,\cdots,2n;31,32,34,\cdots,3n;$ etc., etc.

$$\begin{array}{cccccccccccc} a^1_1A^1_2 &+& a^2_1A^2_2 &+& a^3_1A^3_2 &+& \cdots &+& a^n_1A^n_2 &=& 0\ ,\\ a^1_1A^1_3 &+& a^2_1A^2_3 &+& a^3_1A^3_3 &+& \cdots &+& a^n_1A^n_3 &=& 0\ ,\\ \vdots &&&&&&&&&& \vdots\\ a^1_2A^1_1 &+& a^2_2A^2_1 &+& a^3_2A^3_1 &+& \cdots &+& a^n_2A^n_1 &=& 0\ ,\\ a^1_2A^1_3 &+& a^2_2A^2_3 &+& a^3_2A^3_3 &+& \cdots &+& a^n_2A^n_3 &=& 0\ ,\\ \vdots &&&&&&&&&& \vdots\\ a^1_lA^1_k &+& a^2_lA^2_k &+& a^3_lA^3_k &+& \cdots &+& a^n_lA^n_k &=& 0\ , & (l\neq k).\end{array}$$

A similar set of equations can also be written for the second case when ij are distinct superscripts.

We now show that use of this fundamental property helps us to determine a unique solution of a system of linear equations.

1. Consider the case of two equations in two unknowns:

$$a_1^1 x_1 + a_1^2 x_2 = b^1,$$

$$a_2^1 x_1 + a_2^2 x_2 = b^2.$$

We assume that these two equations are linearly independent. This requires that the determinant of the coefficients $\left|a_j^i\right| \neq 0$. We now proceed to find the non-trivial values of the two unknowns $\{x_1, x_2\}$.

To find the value of the unknown x_1, we multiply the first equation by the $cof\left(a_1^1\right) \equiv A_1^1$ and the second equation by $cof\left(a_2^1\right) \equiv A_2^1$. We thus obtain the two equations:

$$a_1^1 A_1^1 x_1 + a_1^2 A_1^1 x_2 = b^1 A_1^1,$$

$$a_2^1 A_2^1 x_1 + a_2^2 A_2^1 x_2 = b^2 A_2^1.$$

Adding the two equations we find

$$\left(a_1^1 A_1^1 + a_2^1 A_2^1\right) x_1 + \left(a_1^2 A_1^1 + a_2^2 A_2^1\right) x_2 = \left(b^1 A_1^1 + b^2 A_2^1\right).$$

This is written in the form

$$x_1 \sum_{m=1}^{2} a_m^1 A_m^1 + x_2 \sum_{m=1}^{2} a_m^2 A_m^1 = \sum_{m=1}^{2} b^m A_m^1 ,$$

or equivalently

$$x_1 \left(A\delta^{11}\right) + x_2 \left(A\delta^{21}\right) = \sum_{m=1}^{2} b^m A_m^1 .$$

Since $\delta^{21} \equiv 0$ and $\delta^{11} \equiv 1$, this leads to the solution

$$\begin{aligned} x_1 A &= \sum_{m=1}^{2} b^m A_m^1, \\ &= b^1 A_1^1 + b^2 A_2^1 , \\ &= b^1\, cof\left(a_1^1\right) + b^2\, cof\left(a_2^1\right), \\ &= b^1 a_2^2 - b^2 a_1^2 , \\ &= \begin{vmatrix} b^1 & a_1^2 \\ b^2 & a_2^2 \end{vmatrix}. \end{aligned}$$

We now determine the unknown x_2. To find this we multiply the first equation by $cof\left(a_1^2\right) \equiv A_1^2$ and the second by the $cof\left(a_2^2\right) \equiv A_2^2$. We obtain the two equations:

$$a_1^1 A_1^2 x_1 + a_1^2 A_1^2 x_2 = b^1 A_1^2,$$

$$a_2^1 A_2^2 x_1 + a_2^2 A_2^2 x_2 = b^2 A_2^2.$$

Adding the two equations we get

$$\left(a_1^1 A_1^2 + a_2^1 A_2^2\right) x_1 + \left(a_1^2 A_1^2 + a_2^2 A_2^2\right) x_2 = \left(b^1 A_1^2 + b^2 A_2^2\right).$$

We write it in the form

$$x_1 \sum_{m=1}^{2} a_m^1 A_m^2 + x_2 \sum_{m=1}^{2} a_m^2 A_m^2 = \sum_{m=1}^{2} b^m A_m^2,$$

or equivalently

$$x_1 \left(A\delta^{12}\right) + x_2 \left(A\delta^{22}\right) = \sum_{m=1}^{2} b^m A_m^2.$$

Since $\delta^{12} \equiv 0$ and $\delta^{22} \equiv 1$, we are lead to the solution

$$\begin{aligned} x_2 A &= \sum_{m=1}^{2} b^m A_m^2, \\ &= \sum_{m=1}^{2} b^m cof\left(a_m^2\right), \\ &= b^1\, cof\left(a_1^2\right) + b^2\, cof\left(a_2^2\right), \\ &= b^1\left(-a_2^1\right) + b^2 a_1^1, \\ &= \begin{vmatrix} a_1^1 & b^1 \\ a_2^1 & b^2 \end{vmatrix}. \end{aligned}$$

We thus find that the solution of the system of two equations in two unknown in terms of (2×2)-determinants is given by

$$Ax_1 = \begin{vmatrix} b^1 & a_1^2 \\ b^2 & a_2^2 \end{vmatrix},$$

$$Ax_2 = \begin{vmatrix} a_1^1 & b^1 \\ a_2^1 & b^2 \end{vmatrix}.$$

We now invoke our second assumption which requires that the determinant $A \neq 0$. Since

$$A \equiv \begin{vmatrix} a_1^1 & a_1^2 \\ a_2^1 & a_2^2 \end{vmatrix} \neq 0,$$

the solution is unique and is given by

$$x_1 = \frac{\begin{vmatrix} b^1 & a_1^2 \\ b^2 & a_2^2 \end{vmatrix}}{\begin{vmatrix} a_1^1 & a_1^2 \\ a_2^1 & a_2^2 \end{vmatrix}}, \qquad A \neq 0$$

$$x_2 = \frac{\begin{vmatrix} a_1^1 & b^1 \\ a_2^1 & b^2 \end{vmatrix}}{\begin{vmatrix} a_1^1 & a_1^2 \\ a_2^1 & a_2^2 \end{vmatrix}}, \qquad A \neq 0\ .$$

We still have to show that the solution exist. For this we have to show that if the solution exists and is given by the expression above, then when substituted in the original two equations we must have the identities

$$a_1^1 x_1 + a_1^2 x_2 \equiv b^1,$$

$$a_2^1 x_1 + a_2^2 x_2 \equiv b^2.$$

Thus if we substitute the value of x_1 and x_2 in the first equation we get

$$\frac{1}{A}\left\{ a_1^1 \begin{vmatrix} b^1 & a_1^2 \\ b^2 & a_2^2 \end{vmatrix} + a_1^2 \begin{vmatrix} a_1^1 & b^1 \\ a_2^1 & b^2 \end{vmatrix} \right\}$$

$$\equiv \tfrac{1}{A}\left\{ a_1^1\left(b^1 a_2^2 - b^2 a_1^2\right) + a_1^2\left(a_1^1 b^2 - a_2^1 b^1\right)\right\},$$

$$\equiv \tfrac{1}{A}\left\{ b^1\left(a_1^1 a_2^2 - a_2^1 a_1^2\right) + b^2\left(a_1^1 a_1^2 - a_1^2 a_1^1\right)\right\},$$

$$\equiv \tfrac{1}{A}\left\{ b^1\left(a_1^1 a_2^2 - a_2^1 a_1^2\right)\right\},$$

$$\equiv b^1\,, \qquad \text{since } \left(a_1^1 a_2^2 - a_2^1 a_1^2\right) \equiv A.$$

Now substitute the value of x_1 and x_2 in the second equation and we get

$$\tfrac{1}{A}\left\{ a_2^1\left(b^1 a_2^2 - b^2 a_1^2\right) + a_2^2\left(a_1^1 b^2 - a_2^1 b^1\right)\right\}$$

$$\equiv \tfrac{1}{A}\left\{ b^1\left(a_2^1 a_2^2 - a_2^2 a_2^1\right) + b^2\left(a_1^1 a_2^2 - a_2^1 a_1^2\right)\right\},$$

$$\equiv \tfrac{1}{A}\left\{ b^2\left(a_1^1 a_2^2 - a_2^1 a_1^2\right)\right\},$$

$$\equiv b^2\,, \qquad \text{since } \left(a_1^1 a_2^2 - a_2^1 a_1^2\right) \equiv A.$$

Thus the quantities x_1 and x_2 actually represent a solution of the system. We have thus found a procedure for finding the solution which is called Cramer's rule. For finding the solution we thus need to find three determinants A, A_1 and A_2, where

$$A = \begin{vmatrix} a_1^1 & a_1^2 \\ a_2^1 & a_2^2 \end{vmatrix} \neq 0,$$

$$A_1 = \begin{vmatrix} b^1 & a_1^2 \\ b^2 & a_2^2 \end{vmatrix},$$

$$A_2 = \begin{vmatrix} a_1^1 & b^1 \\ a_2^1 & b^2 \end{vmatrix}.$$

Then the unique solution is

$$x_m = \frac{A_m}{A},\ m = 1, 2 \text{ and } A \neq 0.$$

Thus by using Cramer's rule the process of finding the solution of a system of two linear equations has been reduced to finding the quotient of determinants. This procedure is due to CRAMER.

The result

$$Ax_1 = \begin{vmatrix} b^1 & a_1^2 \\ b^2 & a_2^2 \end{vmatrix},$$

$$Ax_2 = \begin{vmatrix} a_1^1 & b^1 \\ a_1^2 & b^2 \end{vmatrix},$$

requires further examination. These two equations are valid whether or not the determinant A is zero or not. When $A \neq 0$, the solution is unique because to determine x_1 and x_2, we can divide by nonzero value of A. On the other hand if the determinant $A = 0$, the system can have a non-trivial solution only when both the determinants on the right-hand side are zero.

Example 1. Using Cramer's rule, find the solution of the system of equations

$$\begin{aligned} 3x - 5y &= 14, \\ x + 2y &= 3. \end{aligned}$$

The determinant of the dyadic coefficients is

$$A \equiv \begin{vmatrix} 3 & -5 \\ 1 & 2 \end{vmatrix} = 11 \neq 0.$$

Hence the non-trivial unique solution is

$$x = \frac{1}{11}\begin{vmatrix} 14 & -5 \\ 3 & 2 \end{vmatrix} = \frac{14 \times 2 + 5 \times 3}{11} = \frac{43}{11},$$

$$y = \frac{1}{11}\begin{vmatrix} 3 & 14 \\ 1 & 3 \end{vmatrix} = \frac{3 \times 3 - 1 \times 14}{11} = -\frac{5}{11}.$$

2. We next consider the case of three equations in three unknowns $\{x_1, x_2, x_3\}$. The three equations are:

$$a_1^1 x_1 + a_1^2 x_2 + a_1^3 x_3 = b^1,$$

$$a_2^1 x_1 + a_2^2 x_2 + a_2^3 x_3 = b^2,$$

$$a_3^1 x_1 + a_3^2 x_2 + a_3^3 x_3 = b^3,$$

where we assume that $Det[a_i^j] \equiv A \neq 0$. Conceptually this problem is very similar to the last problem in two-dimensional space and to avoid repetition, we skip some of the details. Thus to find the unknown x_1, we multiply the first equation by A_1^1, the second equation by A_2^1, the third equation by A_3^1 and then add the three equations. This leads to us to the equation

$$\begin{aligned} & x_1 \left(a_1^1 A_1^1 + a_2^1 A_2^1 + a_3^1 A_3^1\right) \\ + \quad & x_2 \left(a_1^2 A_1^1 + a_2^2 A_2^1 + a_3^2 A_3^1\right) \\ + \quad & x_3 \left(a_1^3 A_1^1 + a_2^3 A_2^1 + a_3^3 A_3^1\right) \\ = \quad & \left(b^1 A_1^1 + b^2 A_2^1 + b^3 A_3^1\right). \end{aligned}$$

Use of the fundamental property of co-factors, tells us that:

$$a^1_1 A^1_1 + a^1_2 A^1_2 + a^1_3 A^1_3 = A,$$

$$a^2_1 A^1_1 + a^2_2 A^1_2 + a^2_3 A^1_3 \equiv 0,$$

$$a^3_1 A^1_1 + a^3_2 A^1_2 + a^3_3 A^1_3 \equiv 0.$$

Hence, we find

$$\begin{aligned} A x_1 &= b^1 A^1_1 + b^2 A^1_2 + b^3 A^1_3, \\ &= b^1 cof(a^1_1) + b^2 cof(a^1_2) + b^3 cof(a^1_3)\ , \\ &= b^1 \begin{vmatrix} a^2_2 & a^3_2 \\ a^2_3 & a^3_3 \end{vmatrix} - b^2 \begin{vmatrix} a^2_1 & a^3_1 \\ a^2_3 & a^3_3 \end{vmatrix} + b^3 \begin{vmatrix} a^2_1 & a^3_1 \\ a^2_2 & a^3_2 \end{vmatrix}, \\ &= \begin{vmatrix} b^1 & a^2_1 & a^3_1 \\ b^2 & a^2_2 & a^3_2 \\ b^3 & a^2_3 & a^3_3 \end{vmatrix}, \end{aligned}$$

where

$$A = \begin{vmatrix} a^1_1 & a^2_1 & a^3_1 \\ a^1_2 & a^2_2 & a^3_2 \\ a^1_3 & a^2_3 & a^3_3 \end{vmatrix}.$$

To find the unknown x_2, we multiply the first equation by A^2_1, second equation by A^2_2 and the the third equation by A^2_3. Then we add the three equations and we obtain

$$\begin{aligned} & x_1 \left(a^1_1 A^2_1 + a^1_2 A^2_2 + a^1_3 A^2_3\right) \\ + \quad & x_2 \left(a^2_1 A^2_1 + a^2_2 A^2_2 + a^2_3 A^2_3\right) \\ + \quad & x_3 \left(a^3_1 A^2_1 + a^3_2 A^2_2 + a^3_3 A^2_3\right) \\ = \quad & b^1 A^2_1 + b^2 A^2_2 + b^3 A^2_3\ . \end{aligned}$$

In this equation the coefficient of x_1 and the coefficient of x_3 are both identically zero, and the coefficient of x_2 is the non-trivial determinant A. We are thus lead to

$$\begin{aligned} x_2 A &= +b^1 A^2_1 + b^2 A^2_2 + b^3 A^2_3\ , \\ &= -b^1 \begin{vmatrix} a^1_2 & a^3_2 \\ a^1_3 & a^3_3 \end{vmatrix} + b^2 \begin{vmatrix} a^1_1 & a^3_1 \\ a^1_3 & a^3_3 \end{vmatrix} - b^3 \begin{vmatrix} a^1_1 & a^3_1 \\ a^1_2 & a^3_2 \end{vmatrix}, \\ &= \begin{vmatrix} a^1_1 & b^1 & a^3_1 \\ a^1_2 & b^2 & a^3_2 \\ a^1_3 & b^3 & a^3_3 \end{vmatrix}. \end{aligned}$$

One can similarly show that

$$x_3 A = \begin{vmatrix} a^1_1 & a^2_1 & b^1 \\ a^1_2 & a^2_2 & b^2 \\ a^1_3 & a^2_3 & b^3 \end{vmatrix}.$$

Thus using Cramer's rule, the solution in terms of the ratio of determinants is

$$x_1 = \frac{1}{A}\begin{vmatrix} b^1 & a_1^2 & a_1^3 \\ b^2 & a_2^2 & a_2^3 \\ b^3 & a_3^2 & a_3^3 \end{vmatrix},$$

$$x_2 = \frac{1}{A}\begin{vmatrix} a_1^1 & b^1 & a_1^3 \\ a_2^1 & b^2 & a_2^3 \\ a_3^1 & b^3 & a_3^3 \end{vmatrix},$$

$$x_3 = \frac{1}{A}\begin{vmatrix} a_1^1 & a_1^2 & b^1 \\ a_2^1 & a_2^2 & b^2 \\ a_3^1 & a_3^2 & b^3 \end{vmatrix},$$

where the determinant $A \neq 0$.

3. Consider next the case of four equations in four unknowns $\{x_1, x_2, x_3, x_4\}$. The four equations are

$$a_1^1x_1 + a_1^2x_2 + a_1^3x_3 + a_1^4x_4 = b^1,$$

$$a_2^1x_1 + a_2^2x_2 + a_2^3x_3 + a_2^4x_4 = b^2,$$

$$a_3^1x_1 + a_3^2x_2 + a_3^3x_3 + a_3^4x_4 = b^3,$$

$$a_4^1x_1 + a_4^2x_2 + a_4^3x_3 + a_4^4x_4 = b^4,$$

where we assume that $Det\left(a_i^j\right) \equiv A \neq 0$.

Without further ado, using Cramer's rule the solution in terms of the ratio of determinants is

$$x_1 = \frac{1}{A}\begin{vmatrix} b^1 & a_1^2 & a_1^3 & a_1^4 \\ b^2 & a_2^2 & a_2^3 & a_2^4 \\ b^3 & a_3^2 & a_3^3 & a_3^4 \\ b^4 & a_4^2 & a_4^3 & a_4^4 \end{vmatrix},$$

$$x_2 = \frac{1}{A}\begin{vmatrix} a_1^1 & b^1 & a_1^3 & a_1^4 \\ a_2^1 & b^2 & a_2^3 & a_2^4 \\ a_3^1 & b^3 & a_3^3 & a_3^4 \\ a_4^1 & b^4 & a_4^3 & a_4^4 \end{vmatrix},$$

$$x_3 = \frac{1}{A}\begin{vmatrix} a_1^1 & a_1^2 & b^1 & a_1^4 \\ a_2^1 & a_2^2 & b^2 & a_2^4 \\ a_3^1 & a_3^2 & b^3 & a_3^4 \\ a_4^1 & a_4^2 & b^4 & a_4^4 \end{vmatrix},$$

$$x_4 = \frac{1}{A}\begin{vmatrix} a_1^1 & a_1^2 & a_1^3 & b^1 \\ a_2^1 & a_2^2 & a_2^3 & b^2 \\ a_3^1 & a_3^2 & a_3^3 & b^3 \\ a_4^1 & a_4^2 & a_4^3 & b^4 \end{vmatrix}.$$

We now state the general result for n linear equations in n unknowns.

THEOREM. Consider a system of n linear equations in n unknowns:

$$\begin{aligned} a_1^1 x_1 + a_1^2 x_2 + \cdots + a_1^n x_n &= b^1, \\ a_2^1 x_1 + a_2^2 x_2 + \cdots + a_2^n x_n &= b^2, \\ &\vdots \\ a_n^1 x_1 + a_n^2 x_2 + \cdots + a_n^n x_n &= b^n, \end{aligned}$$

and let

$$\begin{vmatrix} a_1^1 & a_1^2 & \cdots & a_1^n \\ a_2^1 & a_2^2 & \cdots & a_2^n \\ \vdots & & & \vdots \\ a_n^1 & a_n^2 & \cdots & a_n^n \end{vmatrix} \equiv A \neq 0.$$

Then there exists a unique solution, which in terms of the ratio of two $(n \times n)$-determinants can be written in the form

$$x_k = \frac{\begin{vmatrix} a_1^1 & a_1^2 & \cdots & b^1 & \cdots & a_1^n \\ a_2^1 & a_2^2 & \cdots & b^2 & \cdots & a_2^n \\ \vdots & & & & & \\ a_n^1 & a_n^2 & \cdots & b^n & \cdots & a_n^n \end{vmatrix}}{\begin{vmatrix} a_1^1 & a_1^2 & \cdots & a_1^k & \cdots & a_1^n \\ a_2^1 & a_2^2 & \cdots & a_2^k & \cdots & a_2^n \\ \vdots & & & & & \\ a_n^1 & a_n^2 & \cdots & a_n^k & \cdots & a_n^n \end{vmatrix}}, \qquad k = 1, 2, \cdots, n.$$

When the solution is written in this form, it is called the Cramer's form in terms of determinants.

Example 2. Consider the system of equations

$$2x_1 + 2x_2 - x_3 = 2,$$

$$-3x_1 - x_2 + 3x_3 = -2,$$

$$4x_1 + 2x_2 - 3x_3 = 0.$$

The determinant of the coefficient is

$$\begin{vmatrix} 2 & 2 & -1 \\ -3 & -1 & 3 \\ 4 & 2 & -3 \end{vmatrix} = 2.$$

Therefore using Cramer's rule, the solution is

$$x_1 = \frac{1}{2}\begin{vmatrix} 2 & 2 & -1 \\ -2 & -1 & 3 \\ 0 & 2 & -3 \end{vmatrix} = -\frac{14}{2} = -7 \ ,$$

$$x_2 = \frac{1}{2}\begin{vmatrix} 2 & 2 & -1 \\ -3 & -2 & 3 \\ 4 & 0 & -3 \end{vmatrix} = \frac{10}{2} = 5 \ ,$$

$$x_3 = \frac{1}{2}\begin{vmatrix} 2 & 2 & 2 \\ -3 & -1 & -2 \\ 4 & 2 & 0 \end{vmatrix} = -\frac{12}{2} = -6 \ .$$

Exercises: Using determinants find the solution of the equations

(a)

$$\begin{aligned} 4x - 3y + 2z + 4 &= 0, \\ 6x - 2y + 3z + 1 &= 0, \\ 5x - 3y + 2z + 3 &= 0. \end{aligned}$$

(b)

$$\begin{aligned} 2x + 3y + 5z &= 10, \\ 3x + 7y + 4z &= 3, \\ x + 2y + 2z &= 3. \end{aligned}$$

(c)

$$\begin{aligned} x + y + z &= a\ , \\ x + \epsilon y + \epsilon z &= b\ , \\ x + \epsilon^2 y + \epsilon^2 z &= c\ , \end{aligned}$$

where $\epsilon = \frac{1}{2}(-1 \pm i\sqrt{3}), \epsilon^3 = 1$.

(d)

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 1 & 2 & 3 & 4 \\ 2 & 2 & 1 & 2 & 3 \\ 2 & 2 & 2 & 1 & 2 \\ 2 & 2 & 2 & 2 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 13 \\ 10 \\ 11 \\ 6 \\ 3 \end{pmatrix} .$$

Use (2×2)-minors to find the expansion of determinants which enter in the solution of this problem .

$\cdots\cdots\cdots\cdots$ 08.20.00 $\cdots\cdots\cdots\cdots$