MATRIX ALGEBRA

A **matrix** is a rectangular array of numbers. The numbers in the array are called **entries** in the matrix. The **order** or **dimension** or **size** of a matrix is described by specifying the **number of rows** and the **number of columns**.

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 0 \\ -1 & 4 \end{bmatrix}$$ Here A is a matrix of order 3 × 2, ie, 3 rows and 2 columns.

$\begin{bmatrix} 2 & 1 & 0 & -3 \end{bmatrix}$ is a (1 × 4) – matrix.

$$\text{Let } A = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{bmatrix}$$

Here A is a matrix of **order m × n**. If m = n, then we say that A is a **square** matrix. Thus a square matrix is a matrix with **equal numbers of columns and rows**.

Notation: We write $A = [a_{ij}]_{mxn}$ or simply $[a_{ij}]$.

a_{ij} – is the **entry** (or element) in row i and column j.

Row vector: a matrix of order 1 × n

Column vector: a matrix of order m × 1.

Eg: $\begin{bmatrix} 5 & 0 & 2 & 5 & -1 \end{bmatrix}$; $\begin{bmatrix} 5 \\ 2 \\ 3 \\ -4 \end{bmatrix}$ 4 × 1

1 × 5

- A **zero** matrix is a matrix whose entries are all zero.

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$ Zero matrix of order 3 × 2

- Two matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ are **equal** if they have the **same** order and the **corresponding entries are equal**. **Notation: A = B.**

- Consider $A = \begin{bmatrix} 2 & 1 \\ 3 & x \end{bmatrix}$, $B = \begin{bmatrix} 2 & 1 \\ 3 & 5 \end{bmatrix}$ and $C = \begin{bmatrix} 2 & 1 & 0 \\ 3 & 4 & 0 \end{bmatrix}$. If x = 5 then A = B. For all other values of x, A ≠ B. A and C have **different** sizes. Therefore they are not equal matrices.

MATRIX OPERATIONS (§ 8.5)

THE SUM OF MATRICES

Suppose A and B are matrices of order m × n. Let A = $[a_{ij}]$ and B = $[b_{ij}]$.

The **sum** A + B is a matrix obtained by **adding the corresponding** entries.
Therefore, A + B = C = $[c_{ij}]$ where $c_{ij} = a_{ij} + b_{ij}$

- The order of A + B is also m × n.
- Two matrices **cannot be added** if their **sizes** are **not the same**.

Exercise: Read the examples from the text .

MULTIPLICATION BY A SCALAR

Suppose A = $[a_{ij}]$ is an m × n matrix. Let k be a real number (**scalar**) then **kA is a matrix** whose (i, j)th element is ka_{ij}

Example:

$$A = \begin{bmatrix} 2 & 3 & 4 \\ 1 & -3 & 1 \end{bmatrix}$$

$$3A = \begin{bmatrix} 6 & 9 & 12 \\ 3 & -9 & 3 \end{bmatrix}$$

$$(-1)A = \begin{bmatrix} -2 & -3 & -4 \\ -1 & 3 & -1 \end{bmatrix}$$

$$\tfrac{1}{2}A = \begin{bmatrix} 1 & \tfrac{3}{2} & 2 \\ \tfrac{1}{2} & \tfrac{-3}{2} & \tfrac{1}{2} \end{bmatrix}$$

NOTATION: -A means (-1)A
A-B means A + (-1)B

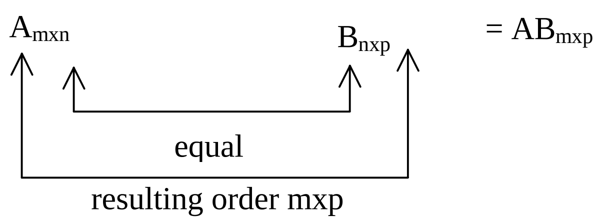
MATRIX MULTIPLICATION

Suppose A = $[a_{ij}]$ is an m×n matrix and B = $[b_{ij}]$ is an n×p matrix.

The **product AB** is defined as a matrix C = $[c_{ij}]$ of order m×p where $c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$

- The $(i,j)^{th}$ – entry of the matrix C (= AB) is computed using the i^{th} row $[a_{i1} \;\; a_{i2} \;\; \ldots \;\; a_{in}]$ of A and the j^{th} column $\begin{bmatrix} b_{1j} \\ b_{2j} \\ \ldots \\ b_{nj} \end{bmatrix}$ of B.

- The product AB of two matrices A and B is **only defined if the number of columns in A = the number of rows in B.**



- The order of AB is $m \times p$.

Example

Suppose $A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 6 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 1 & 4 & 3 \\ 0 & -1 & 3 & 1 \\ 2 & 7 & 5 & 2 \end{bmatrix}$

A is a 2×3 matrix; B is a 3×4 matrix. Therefore AB is a 2×4 matrix.

$$AB = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 6 & 0 \end{bmatrix} \begin{bmatrix} 4 & 1 & 4 & 3 \\ 0 & -1 & 3 & 1 \\ 2 & 7 & 5 & 2 \end{bmatrix} = \begin{bmatrix} . & . & . & . \\ . & . & 26 & . \end{bmatrix}$$

The (2,3)-entry $= (2 \times 4) + (6 \times 3) + (0 \times 5) = 26$.

The entry in row 1 and column 4 of AB is given by

$(1 \times 3) + (2 \times 1) + (4 \times 2) = 13$ etc.

Thus $AB = \begin{bmatrix} 12 & 27 & 30 & 13 \\ 8 & -4 & 26 & 12 \end{bmatrix}$ (Verify). Here BA is **not** defined

Remark: Even when AB and BA are both defined for matrices A and B, **they need not be equal.**

For example, $A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$ $B = \begin{bmatrix} 2 & -1 \\ 3 & 0 \end{bmatrix}$ $AB = \begin{bmatrix} 14 & -1 \\ 13 & -2 \end{bmatrix}$ *and* $BA = \begin{bmatrix} 0 & 5 \\ 3 & 12 \end{bmatrix}$

PROPERTIES OF MATRIX OPERATIONS

Assume that the orders of the matrices are such that the following operations are defined.

1. A + B = B + A (addition is commutative)
2. A + (B + C) = (A + B) + C (addition is associative)
3. A(BC) = (AB)C (multiplication is associative)
4. (A + B)C = AC + BC (distributive)
5. k (A + B) = kA = kB

etc.

Read the text; pp 661-666

TRANSPOSE OF A MATRIX

Suppose A = $[a_{ij}]$ is an mxn matrix. Then the **transpose A^T** of A is the n × m matrix defined as

$$A^T = [a'_{ij}] \quad \text{where}$$

$$a'_{ij} = a_{ji} \text{ (the (i,j)-entry of } A^T \text{ is the (j,i)-entry of A)}$$

i.e. the rows of A^T are the columns of A.

e.g. Let $A = \begin{bmatrix} 2 & 3 & 1 \\ 4 & 2 & 3 \end{bmatrix}$ be a (2 × 3)-matrix. Then

$A^T = \begin{bmatrix} 2 & 4 \\ 3 & 2 \\ 1 & 3 \end{bmatrix}$. Its dimension is 3 × 2, i.e., it has 3 rows and 2 columns.

Def: A square matrix A is said to be a **symmetric matrix** if **A^T = A,** and a **skew-symmetric matrix** if **A^T = -A.**

PROPERTIES OF THE TRANSPOSE

(a) $(A^T)^T = A$
(b) $(A + B)^T = A^T + B^T$
(c) $(kA)^T = kA^T$, for any scalar k
(d) $(AB)^T = B^TA^T$

Proof of (d): Suppose $A = [a_{ij}]_{m\times n}$, $B = [b_{ij}]_{n\times p}$ and $C = [c_{ij}]_{m\times p}$ such that C = AB. Then $c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$.

$C^T = [c'_{ij}]$ where $c'_{ij} = c_{ji}$. The (i,j) entry in $C^T = c'_{ij} = c_{ji} = \sum_{k=1}^{n} a_{jk} b_{ki}$.

Now the (i,j) entry in $B^T A^T$ $= \sum_{k=1}^{n} b'_{ik} a'_{kj}$

$$= \sum_{k=1}^{n} b_{ki} a_{jk}$$

= (i,j) entry in C^T.

$\therefore\ C^T = B^T A^T$

i.e. $(AB)^T = B^T A^T$.

Definition: Suppose A is a square matrix. (i.e. number of rows of A = number of columns of A = n, say).

$A = [a_{ij}]_{n\times n}$

The entries $a_{11}, a_{22}, \ldots, a_{nn}$ are said to be on the **main diagonal of A.**

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

main diagonal

Notation: $A^2 = A \cdot A$

$A^m = \underbrace{A \cdot A \ldots A}_{\text{m factors}}$

Definition: An **identity matrix** is a square matrix with 1's on the main diagonal and 0's elsewhere.

- **I_n:** identity matrix of order $n \times n$.
- $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Theorem: For any m × n matrix A, we have $AI_n = I_mA = A$.

Definition: Suppose A and B are square matrices.

If AB = BA = I, the identity matrix, then A is said to be **invertible** and B is called an **inverse** of A.

Note: If B is an inverse of A, then B is invertible and A is an inverse of B.

Example:

$$A = \begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix} \qquad B = \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix}$$

$$AB = \begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 6-5 & 10-10 \\ -3+3 & -5+6 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$BA = \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Here A, B are invertible.

Note that not all matrices are invertible.

Suppose $A = \begin{bmatrix} 1 & 4 & 0 \\ 2 & 5 & 0 \\ 3 & 6 & 0 \end{bmatrix}$.

We will show that A is not invertible.

Suppose $B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$ is a 3 × 3 matrix. Consider BA. It is easy to check that the 3rd column of BA is $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. ∴ BA ≠ I.

Thus there is no matrix B such that BA = I. Hence the matrix A is not invertible.

THEOREM 1: If a matrix is invertible then its inverse is **unique**.

Proof: Suppose A is invertible. Let B and C be inverses of A.

∴ $$AB = BA = I \qquad (1)$$

and $$AC = CA = I \qquad (2)$$

We will show that B = C.

Since AB = I, C(AB) = CI = C (3)

But C(AB) = (CA)B = IB = B (4)

From (3) and (4) we have B = C.

$\therefore$ The inverse of A is **unique**.

NOTATION: The inverse of A is denoted as A^{-1}.

Therefore $A A^{-1} = A^{-1}A = I$.

Note that the inverse of A^{-1} is A, that is, $(A^{-1})^{-1} = A$.

Result: If A and B are invertible matrices of the same order then the inverse of AB is $B^{-1}A^{-1}$.

Proof: $(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1}$
$= (AI)A^{-1}$
$= AA^{-1} = I$

Similarly we can show that $(B^{-1}A^{-1})(AB) = I$.

Therefore, AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$.

SYSTEMS OF LINEAR EQUATIONS (§ 8.4)

The equation

$$\alpha_1 x_1 + \alpha_2 x_2 + \ldots + \alpha_n x_n = \beta \qquad \text{-------------} \qquad (*)$$

is a **linear equation** in n variables $x_1, x_2, \ldots, x_n$.

Here α_i's and β are real numbers.

A collection of linear equations in $x_1, \ldots, x_n$ is referred to as a **system of linear equations**.

For example,

$$2x_1 + 3x_2 + 5x_3 = 7$$
$$4x_1 + x_2 - 3x_3 = 6$$

is a system of **two** equations in **three** variables (unknowns) x_1, x_2 and x_3.

A sequence of numbers $s_1, s_2, \ldots, s_n$ is a **solution to the system** (*) if

$$\alpha_1 s_1 + \alpha_2 s_2 + \ldots + \alpha_n s_n = \beta.$$

Example: Consider the system of equations:

$$x + y + z = 3$$
$$2x + y + 3z = 1$$

Clearly $\left.\begin{array}{r} x=-2 \\ y=5 \\ z=0 \end{array}\right\}$ and $\left.\begin{array}{r} x=0 \\ y=4 \\ z=-1 \end{array}\right\}$ are solutions to the above system.

Consider the system

$$\begin{array}{ccccc} a_{11}x_1 + & a_{12}x_2 + & \ldots + & a_{1n}x_n = & b_1 \\ a_{21}x_1 + & a_{22}x_2 + & \ldots + & a_{2n}x_n = & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1}x_1 + & a_{m2}x_2 + & \ldots + & a_{mn}x_n = & b_m \end{array} \quad \text{-----------(1)}$$

This system is constructed using the rectangular array

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \ldots & a_{1n} & b_1 \\ a_{21} & a_{22} & \ldots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} & b_n \end{array}\right]$$

This $m \times (n + 1)$-matrix is called the **augmented matrix** for the system (1). To solve the system (1) we **eliminate** the variables.

Example: Solve the system given below:

$$\begin{array}{llll} 2x & + y & - 2z & = 10 \quad \text{-----(1)} \\ 3x & + 2y & + 2z & = 1 \quad \text{-----(2)} \\ 5x & + 4y & + 3z & = 4 \quad \text{-----(3)} \end{array}$$

<u>ELEMENTARY ROW OPERATIONS (e.r.o.)</u> (§ 8.4)

An **elementary row operation (e.r.o)** is an operation performed on the rows of a matrix and is one of **3 types.**

1. **Multiply a row by a non-zero constant;**
2. **Interchange any two rows;**
3. **Add a multiple of one row to another row.**

By these operations we can transform the given matrix into the so called **row echelon form**. To solve a system of equations we transform the augmented matrix into **row echelon form**. This form facilitates the solving for the unknown variables.

DEFINITION:

- The **leading entry** of a row in a matrix is the **first non-zero** entry in the row.

- A matrix is in **row-echelon form** if the **leading** entry in **each row** (except the first) is to **the right** of the leading entry of the **preceding row**. The leading entry in each row is 1. All rows consisting of zeros are at the bottom of the matrix.
- A matrix is in **reduced row echelon form** if, in addition, every number above and below each leading entry is a 0.

Examples: Consider the matrices given below: The first one is in row echelon form; the second matrix is in reduced row echelon form; the third matrix is not in row echelon form.

$$\begin{bmatrix} \textcircled{1} & 2 & 0 & 5 & -6 \\ 0 & \textcircled{1} & 1 & 2 & 0 \\ 0 & 0 & 0 & \textcircled{1} & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The circled entries are the leading entries.

$$\begin{bmatrix} 0 & \textcircled{1} & 0 & 0 & 4 & 0 \\ 0 & 0 & \textcircled{1} & 0 & -3 & 0 \\ 0 & 0 & 0 & \textcircled{1} & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & \textcircled{1} \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & -0.5 & 0 \\ 1 & 0 & 3 & 4 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

DEFINITION: Two matrices A and B are **row-equivalent** when one can be obtained from the other using elementary row operations.
Notation: A ~ B

THE RANK OF A MATRIX

DEFINITION: A set of row vectors $\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_n$ is a **linearly dependent** set if there are scalars, $c_1, c_2, \ldots, c_n$, **not all zero**, such that:

$$c_1 \cdot \mathbf{r}_1 + c_2 \cdot \mathbf{r}_2 + \ldots + c_n \cdot \mathbf{r}_n = 0. \qquad \text{-------------}(*)$$

The set of vectors $\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_n$ is **linearly independent** if the equation (*) implies that:

$$c_1 = c_2 = \ldots = c_n = 0.$$

DEFINITION: The **rank** of a matrix A is the **maximum number of linearly independent rows of A.**

Notation: r(A).

Theorem: The rank of a matrix A, **in row echelon form**, is the number of non-zero rows of A.

Theorem: If A and B are row equivalent, then r(A) = r(B).

TYPES OF SOLUTIONS OF A SYSTEM OF LINEAR EQUATIONS (§ 8.4)

Consider the system:

$$\begin{array}{ccccc} a_{11}x_1 + & a_{12}x_2 + & \ldots + & a_{1n}x_n = & b_1 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1}x_1 + & a_{m2}x_2 + & \ldots + & a_{mn}x_n = & b_n \end{array} \qquad \text{----------------------- (1)}$$

Let $A = [a_{ij}]_{m\times n}$, $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ *and* $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$

The system (1) can be rewritten as A**x** = **b** ------------------------ (2)

The augmented matrix for the given system is:

$$[A\,|\,\mathbf{b}] = \left[\begin{array}{cccc|c} a_{11} & a_{12} & \ldots. & a_{1n} & b_1 \\ a_{21} & a_{22} & \ldots. & a_{2n} & b_2 \\ \ldots & & & & \\ a_{m1} & a_{m2} & \ldots. & a_{mn} & b_n \end{array}\right]$$

(i) If $r(A) \neq r(A\,|\,\mathbf{b})$ then the equations in system (1) are **inconsistent and therefore there are no solutions.**

(ii) If $r(A) = r(A\,|\,\mathbf{b}) = n$, the system is **consistent and has a unique solution.**

(iii) If $r(A) = r(A\,|\,\mathbf{b}) = r < n$, then the system is **consistent. It has more than one solution.** In this case there are **n-r arbitrary variables.**

NOTE: When the augmented matrix of a system of linear equations has been reduced to the row-echelon form, the variables corresponding to columns containing a leading entry are called **leading variables**. The non-leading variables are taken as the **arbitrary variables**. These provide the **parameters** in the final solution.

HOMOGENOUS SYSTEM OF LINEAR EQUATIONS

A system of equations of the form

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = 0 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = 0 \\ \dots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = 0 \end{cases} \qquad \text{---------------------------- } (*)$$

is said to be a **homogenous** system of equations.
The system (*) can be rewritten as A **x** = **0**

NOTES:

Clearly $x_1 = x_2 = \dots = x_n = 0$ is **always** a solution of the system (*). It is referred to as the **trivial solution.**

If m < n then the system (*) has **infinitely many non-trivial solutions.**

THEOREM:

Let A be an n × n matrix. The following are equivalent:

(a) A is invertible.
(b) A **x** = 0 has only the trivial solution.
(c) A is **row equivalent** to I.

Definition: A **non-invertible** matrix is said to be **singular**. If a matrix is invertible then it is said to be **non-singular.**