1.1 Linear systems

Examples of linear systems and explanation of the term *linear*.

(1) $ax = b$

(2) $a_1 x_1 + a_2 x_2 + \ldots\ldots + a_n x_n = b$

Illustration by another example:

The equation $2x_1 - 3x_2 + 5x_3 = 7$ has one solution as $x_1 = 4$, $x_2 = 2$ and $x_3 = 1$. However, we can check that $x_1 = 1$, $x_2 = 5$ and $x_3 = 4$ is also a solution to this equation.

A *linear system* is one containing *m* equations in *n* unknowns. It is very conveniently denoted by

$$\begin{array}{cccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \ldots\ldots\ldots + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \ldots\ldots\ldots + & a_{2n}x_n & = & b_2 \\ . & & . & & & . & & . \\ . & & . & & & . & & . \\ . & & . & & & . & & . \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \ldots\ldots\ldots + & a_{mn}x_n & = & b_m \end{array}$$

The general coefficient is denoted by a_{ij} where the first subscript *i* stands for the row number and the second subscript *j* stands for the column number. A solution to the above system is in fact a set of numbers $\{ s_1, s_2, \ldots\ldots, s_n \}$ such that each equation is satisfied when the substitutions $x_1 = s_1$, $x_2 = s_2$, ……., $x_n = s_n$ are carried out.

To find solutions to linear systems, we use the *method of elimination*, that is we eliminate some of the unknowns by *adding a multiple of one equation to another equation*.

The interesting part is that we will not only consider systems with equal number of unknowns and equations but as well as all types of systems, that is, with $m < n$ and $m > n$.

EXAMPLE 1 $(m = n)$

The linear system

$$x - 2y = 7$$
$$2x + 5y = -4$$

can be easily simplified by multiplying the first equation by –2 and adding the result to the second equation. This gives

$$9y = -18 \text{ and, hence, } y = -2 .$$

Substitution of this value in any of the two original equations will yield $x = 3$.

EXAMPLE 2 $(m = n)$

The linear system

$$x - 2y = 7$$
$$2x - 4y = -4$$

can be simplified by the same procedure used in the previous example but this time we end up with the equation

$$0 = -18 \text{ which is absurd!}$$

Thus, the above system has no solutions.

NOTE An alert observer would have seen that the left hand side of the second equation is exactly twice the left hand side of the first equation but that the right hand side of the second equation is not twice the right hand side of the first.

EXAMPLE 3 $(m = n)$

Consider the linear system

$$x + 2y - z = 6$$
$$2x + y + 2z = 5$$
$$4x - 3y + z = 8$$

We multiply the first equation by –2 and add the result to the second equation and then multiply the first equation by –4 and add the result to the third equation. The resulting equations are

$$-3y + 4z = -7$$
$$-11y + 5z = -16$$

We have thus reduced the above system to a new one with only two equations and two unknowns.

Now, if we multiply the first equation by –11 and add it to *three* times the second equation (in order to eliminate *z*), we obtain

$$-29z = 29$$

which means that

$$z = -1.$$

Substituting the value of z in the equation $-3y + 4z = -7$ gives

$$y = 1$$

and, finally, by substituting the values of both y and z in the equation $x + 2y - z = 6$ will give

$$x = 3.$$

In this case, we have a *unique* solution.

NOTE Our elimination procedure has produced the linear system

$$\begin{aligned} x + 2y - z &= 6 \\ -3y + 4z &= -7 \\ z &= -1 \end{aligned}$$

MORE UNKNOWNS THAN EQUATIONS

It is obvious, from the very beginning, that we will never be able to really assign only a value to each unknown, that is, have a unique solution. On the contrary, there is the possibility of having an *infinite* number of solutions.

EXAMPLE 4 $(m < n)$

The system

$$\begin{aligned} x + 2y - z &= 5 \\ 2x - y - z &= -3 \end{aligned}$$

has fewer equations than unknowns.

Multiplying the first equation by –2 and adding the result to the second equation gives

$$-5y + z = -13 \text{ and } z = 5y - 13.$$

Substituting in the first equation, we have

$$x + 2y - 5y + 13 = 5\text{, that is, } x = 3y - 8.$$

Since we cannot have a unique solution, we have to express two of the variables in terms of the third one (in this case, x and z in terms of y). Thus, if y takes on the value r, the general solution of this linear system can be given as

$$x = 3r - 8$$
$$y = r$$
$$z = 5r - 13$$

NOTE The reader can check that these equations always satisfy the first two original equations.

MORE EQUATIONS THAN UNKNOWNS

These are examples where there are more equations than unknowns. It should be very easy to find the solution(s) but we should be very careful about the third equation because it may agree with the first two or simply disagree with them. We will consider both cases.

EXAMPLE 5 $(m > n)$

The linear system

$$x + 3y = 1$$
$$2x - y = 9$$
$$3x + 4y = 8$$

can be simplified as follows:

Multiply the first equation by –2 and add to the second equation. This gives

$$-7y = 7 \text{, that is, } y = -1.$$

Multiplying the first equation by –3 and adding to the third equation gives

$$-5y = 5 \text{, that is, } y = -1.$$

The above system has thus been *reduced* to the following system

$$x + 3y = 1$$
$$y = -1$$
$$y = -1$$

Thus, there exists a unique solution, that is, $y = -1$ and $x = 4$.

NOTE We say that the third equation is *redundant* or that it is *linearly dependent* on the two others. In other words, it is a combination of the other two.

EXAMPLE 6 $(m > n)$

Another example with more equations than unknowns is the system

$$\begin{aligned} x - 2y &= -4 \\ 2x + y &= 7 \\ 5x - y &= -2 \end{aligned}$$

Multiplying the first equation by –2 and adding to the second gives

$$5y = 15\text{, that is, } y = 3.$$

Multiplying the first equation by –5 and adding to the third gives

$$9y = 18\text{, that is, } y = 2.$$

This time, the system has been reduced to

$$\begin{aligned} x - 2y &= -4 \\ y &= 3 \\ y &= 2 \end{aligned}$$

It is clear that there is no solution since y cannot be equal to 2 and 4 at the same time!

In all the above examples, we have seen that the *method of elimination* consists of

(1) interchanging equations,
(2) multiplying an equation by a non-zero constant and
(3) adding a multiple of an equation to another equation.

A linear system with two equations and two unknowns like

$$\begin{aligned} a_1x + a_2y &= c_1 \\ b_1x + b_2y &= c_2 \end{aligned}$$

can be explained graphically. These two equations can be represented by two straight lines l_1 and l_2 respectively. There are three possible cases :

(1) The system has a *unique solution*, that is, the lines intersect.
(2) The system has *no solution*, that is, the lines do not intersect.
(3) The system has an *infinite number of solutions*, that is, the lines coincide.

EXAMPLE 7 (Linear programming)

A company produces three products from the same production process.

Each unit of product A requires 30 minutes in Department I and 30 minutes in Department II.

Each unit of product B requires 30 minutes in Department I and 20 minutes in Department II.

Each unit of product C requires 20 minutes in Department I and 30 minutes in Department II.

In any working day there are 8 hours available in Department I and 9 hours in Department II.

The objective is to determine the number of units of each product to be manufactured in order to try and utilise fully the number of hours available in each Department.

SOLUTION

If x_1 units of product A, x_2 units of product B and x_3 units of product C are manufactured, then the following equations should be satisfied (bearing in mind that the values of x_1, x_2 and x_3 should all be non-negative integers):

$$30x_1 + 30x_2 + 20x_3 = 480, \text{ that is, } 3x_1 + 3x_2 + 2x_3 = 48$$
$$30x_1 + 20x_2 + 40x_3 = 540, \text{ that is, } 3x_1 + 2x_2 + 3x_3 = 54$$

Multiplying the first equation by –1 and adding to the second gives

$$-x_2 + x_3 = 6 \text{ or } x_2 = x_3 - 6$$

Substituting in the first equation, we have

$$3x_1 + 3(x_3 - 6) + 2x_3 = 48$$
$$\Rightarrow x_1 = \frac{66 - 5x_3}{3}$$

Since $x_1 \geq 0$, $x_2 \geq 0$ and $x_3 \geq 0$, it follows that $x_3 \leq 13$.
It can easily be checked that
$x_1 = 4$, $x_2 = 3$ and $x_3 = 9$ or $x_1 = 2$, $x_2 = 6$ and $x_3 = 12$ are two possible solutions. Further restrictions or additional information may develop an *optimum* solution.