Harvey Mudd College Math Tutorial:

Elementary Vector Analysis

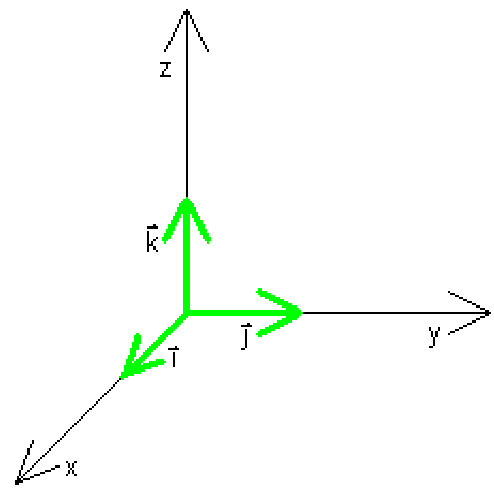
In order to measure many physical quantities, such as force or velocity, we need to determine both a magnitude and a direction. Such quantities are conveniently represented as vectors.

The direction of a vector $\vec{v}$ in 3-space is specified by its components in the x, y, and z directions, respectively:

$$(x, y, z) \quad \text{or} \quad x\vec{i} + y\vec{j} + z\vec{k},$$

where $\vec{i}$, $\vec{j}$, and $\vec{k}$ are the **coordinate vectors** along the x, y, and z-axes.

$\vec{i} = (1, 0, 0)$
$\vec{j} = (0, 1, 0)$
$\vec{k} = (0, 0, 1)$



The magnitude of a vector $\vec{v} = (x, y, z)$, also called its length or **norm**, is given by

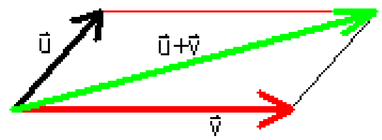
$$\|\vec{v}\| = \sqrt{x^2 + y^2 + z^2}$$

Notes

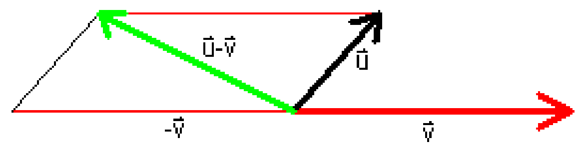
- Vectors can be defined in any number of dimensions, though we focus here only on 3-space.
- When drawing a vector in 3-space, where you position the vector is unimportant; the vector's essential properties are just its magnitude and its direction. Two vectors are **equal** if and only if corresponding components are equal.
- A vector of norm 1 is called a **unit vector**. The coordinate vectors are examples of unit vectors.
- The zero vector, $\vec{0} = (0, 0, 0)$, is the only vector with magnitude 0.

Basic Operations on Vectors

To add or subtract vectors $\vec{u} = (u_1, u_2, u_3)$ and $\vec{v} = (v_1, v_2, v_3)$, add or subract the corresponding coordinates:

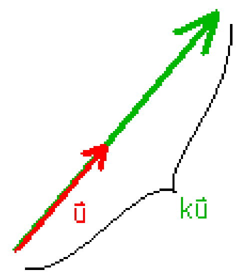


$$\begin{aligned} \vec{u} + \vec{v} &= (u_1 + v_1, u_2 + v_2, u_3 + v_3) \\ \vec{u} - \vec{v} &= (u_1 - v_1, u_2 - v_2, u_3 - v_3) \end{aligned}$$



To multiply vector $\vec{u}$ by a scalar k, multiply each coordinate of $\vec{u}$ by k:

$$k\vec{u} = (ku_1, ku_2, ku_3)$$



Example

The vector $\vec{v} = (2, 1, -2) = 2\vec{i} + \vec{j} - 2\vec{k}$ has magnitude

$$\|\vec{v}\| = \sqrt{2^2 + 1^2 - (-2)^2} = 3.$$

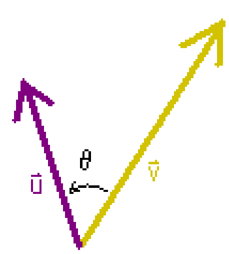
Thus, the vector $\frac{1}{3}\vec{v} = \left(\frac{2}{3}, \frac{1}{3}, \frac{-2}{3}\right)$ is a unit vector in the same direction as $\vec{v}$.

In general, for $\vec{v} \neq \vec{0}$, we can scale (or **normalize**) $\vec{v}$ to the unit vector as $\frac{\vec{v}}{\|\vec{v}\|}$ pointing in the same direction as $\vec{v}$.

Dot Product

Let $\vec{u} = (u_1, u_2, u_3)$ and $\vec{v} = (v_1, v_2, v_3)$. The **dot product** $\vec{u} \cdot \vec{v}$ (also called the **scalar product** or **Euclidean inner product**) of $\vec{u}$ and $\vec{v}$ is defined in two distinct (though equivalent) ways:

$$\begin{aligned} \vec{u} \cdot \vec{v} &= u_1 v_1 + u_2 v_2 + u_3 v_3 \\ &= \begin{cases} \|\vec{u}\| \, \|\vec{v}\| \cos\theta & \text{if } \vec{u} \neq \vec{0}, \vec{v} \neq \vec{0} \\ 0 & \text{if } \vec{u} = \vec{0} \text{ or } \vec{v} = \vec{0} \end{cases} \\ &\quad \text{where } 0 \leq \theta \leq \pi \text{ is the angle between } \vec{u} \text{ and } \vec{v} \end{aligned}$$



Why are the two definitions equivalent?

Properties of the Dot Product

- $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$
- $\vec{u} \cdot (\vec{v} + \vec{w}) = (\vec{u} \cdot \vec{v}) + (\vec{u} \cdot \vec{w})$
- $\vec{u} \cdot \vec{u} = \|\vec{u}\|^2$

See if you can verify each of these!

Example

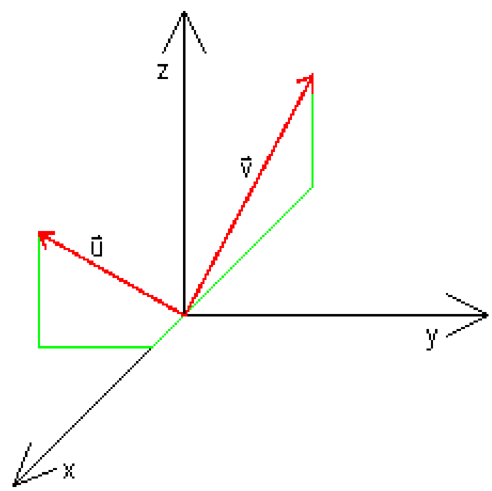
If $\vec{u} = (1, -2, 2)$ and $\vec{v} = (-4, 0, 2)$, then

$$\begin{aligned} \vec{u} \cdot \vec{v} &= (1)(-4) + (-2)(0) + (2)(2) \\ &= -1 + 0 + 4 \\ &= 0 \end{aligned}$$

Using the second definition of the dot product with $\|\vec{u}\| = 3$ and $\|\vec{v}\| = 2\sqrt{5}$,

$$\vec{u} \cdot \vec{v} = 0 = 6\sqrt{5} \cos\theta$$

so $\cos\theta = 0$, yielding $\theta = \frac{\pi}{2}$.
Though we might not have guessed it, $\vec{u}$ and $\vec{v}$ are perpendicular to each other!
In general,



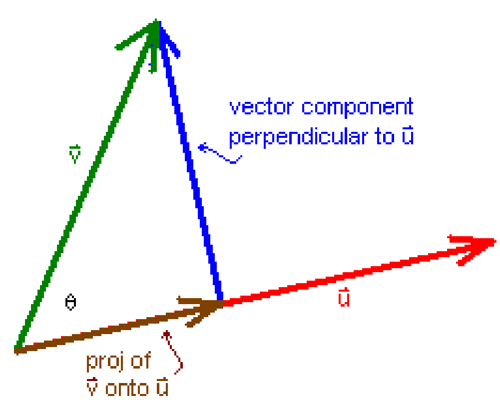
> Two non-zero vectors $\vec{u}$ and $\vec{v}$ are perpendicular (or **orthonormal** if and only if $\vec{u} \cdot \vec{v} = 0$.

Proof

Projection of a Vector

It is often useful to resolve a vector $\vec{v}$ into the sum of vector components parallel and perpendicular to a vector $\vec{u}$.
Consider first the parallel component, which is called the **projection of $\vec{v}$ onto $\vec{u}$**. This projection should be in the direction of $\vec{u}$ and should have magnitude $\|\vec{v}\| \cos\theta$, where $0 \leq \theta \leq \pi$ is the angle between $\vec{u}$ and $\vec{v}$. Let's normalize $\vec{u}$ to $\frac{\vec{u}}{\|\vec{u}\|}$ and then scale this by the magnitude $\|\vec{v}\| \cos\theta$:

$$\begin{aligned} \text{projection of } \vec{v} \text{ onto } \vec{u} &= (\|\vec{v}\| \cos\theta) \frac{\vec{u}}{\|\vec{u}\|} \\ &= \frac{\|\vec{v}\|\|\vec{u}\| \cos\theta}{\|\vec{u}\|^2} \vec{u} \\ &= \frac{\vec{v} \cdot \vec{u}}{\|\vec{u}\|^2} \vec{u} \end{aligned}$$



The perpendicular vector component of $\vec{v}$ is then just the difference between $\vec{v}$ and the projection of $\vec{v}$ onto $\vec{u}$.

In summary,

projection of $\vec{v}$ onto $\vec{u}$	$=$	$\frac{\vec{v}\cdot\vec{u}}{\|\vec{u}\|^2}\vec{u}$
vector component of $\vec{v}$ perpendicular to $\vec{u}$	$=$	$\vec{v} - \frac{\vec{v}\cdot\vec{u}}{\|\vec{u}\|^2}\vec{u}$

Cross Product

Let $\vec{u} = (u_1, u_2, u_3)$ and $\vec{v} = (v_1, v_2, v_3)$. The **cross product** $\vec{u} \times \vec{v}$ yields a vector perpendicular to both $\vec{u}$ and $\vec{v}$ with direction determined by the right-hand rule. Specifically,

$$\vec{u} \times \vec{v} = (u_2v_3 - u_3v_2)\vec{i} - (u_1v_3 - u_3v_1)\vec{j} + (u_1v_2 - u_2v_1)\vec{k}$$

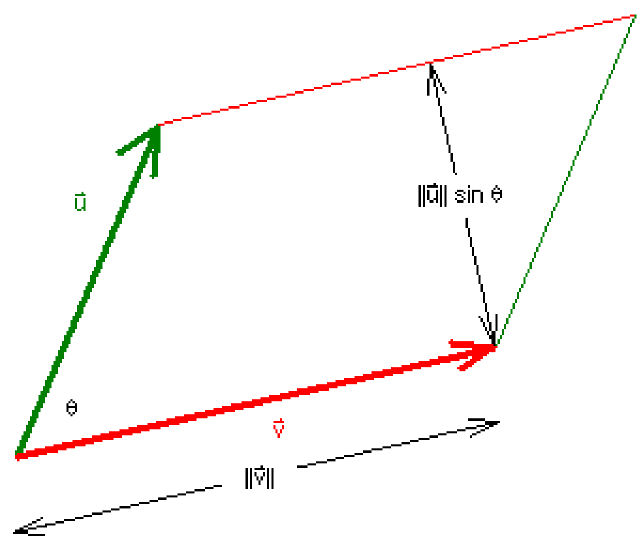
It can also be shown that

$$\|\vec{u} \times \vec{v}\| = \|\vec{u}\|\,\|\vec{v}\| \sin\theta \quad \text{for } \vec{u} \neq \vec{0}, \quad \vec{v} \neq \vec{0}$$

where $0 \leq \theta \leq \pi$ is the angle between $\vec{u}$ and $\vec{v}$.

Proof

Thus, the magnitude $\|\vec{u} \times \vec{v}\|$ gives the area of the parallelogram formed by $\vec{u}$ and $\vec{v}$.
As implied by the geometric interpretation,



Non zero vectors $\vec{u}$ and $\vec{v}$ are parallel if and only if $\vec{u} \times \vec{v} = \vec{0}$.

Proof

Properties of the Cross Product

- $\vec{u} \times \vec{v} = -(\vec{v} \times \vec{u})$
- $\vec{u} \times (\vec{v} + \vec{w}) = (\vec{u} \times \vec{v}) + (\vec{u} \times \vec{w})$
- $\vec{u} \times \vec{u} = \vec{0}$

Again, see if you can verify each of these.

Connections between the Dot Product and Cross Product

In the following Exploration, select values for the components of $\vec{u}$ and $\vec{v}$. You will see $\vec{u} \cdot \vec{v}$ and $\vec{u} \times \vec{v}$ computed and $\vec{u}$, $\vec{v}$, and $\vec{u} \times \vec{v}$ displayed on a coordinate system.

Exploration

Key Concepts

Let $\vec{u} = (u_1, u_2, u_3)$ and $\vec{v} = (v_1, v_2, v_3)$.

- **Basic Operations, Norm of a vector**

$$\begin{aligned} \vec{u} + \vec{v} &= (u_1 + v_1, u_2 + v_2, u_3 + v_3) \\ \vec{u} - \vec{v} &= (u_1 - v_1, u_2 - v_2, u_3 - v_3) \\ k\vec{u} &= (ku_1, ku_2, ku_3) \\ \|\vec{v}\| &= \sqrt{x^2 + y^2 + z^2} \end{aligned}$$

- **Dot Product**

$$\begin{aligned} \vec{u} \cdot \vec{v} &= u_1 v_1 + u_2 v_2 + u_3 v_3 \\ &= \begin{cases} \|\vec{u}\| \, \|\vec{v}\| \cos\theta & \text{if } \vec{u} \neq \vec{0}, \vec{v} \neq \vec{0} \\ 0 & \text{if } \vec{u} = \vec{0} \text{ or } \vec{v} = \vec{0} \end{cases} \\ & \qquad \text{where } 0 \leq \theta \leq \pi \text{ is the angle between } \vec{u} \text{ and } \vec{v} \end{aligned}$$

  for $\vec{u} \neq \vec{0}, \quad \vec{v} \neq \vec{0}$,

  $\vec{u} \cdot \vec{v} = 0$ if and only if $\vec{u}$ is orthogonal to $\vec{v}$.

- **Projection of a Vector**

  projection of $\vec{v}$ onto $\vec{u}$ $= \frac{\vec{v} \cdot \vec{u}}{\|\vec{u}\|^2} \vec{u}$

  vector component of $\vec{v}$ perpendicular to $\vec{u}$ $= \vec{v} - \frac{\vec{v} \cdot \vec{u}}{\|\vec{u}\|^2} \vec{u}$

- **Cross Product**

$$\begin{aligned} \vec{u} \times \vec{v} &= (u_2 v_3 - u_3 v_2)\vec{i} - (u_1 v_3 - u_3 v_1)\vec{j} + (u_1 v_2 - u_2 v_1)\vec{k} \\ \|\vec{u} \times \vec{v}\| &= \|\vec{u}\| \, \|\vec{v}\| \sin\theta \quad \text{for } \vec{u} \neq \vec{0}, \quad \vec{v} \neq \vec{0} \\ & \qquad \text{where } 0 \leq \theta \leq \pi \text{ is the angle between } \vec{u} \text{ and } \vec{v}. \end{aligned}$$

[I'm ready to take the quiz.] [I need to review more.]
[Take me back to the Tutorial Page]