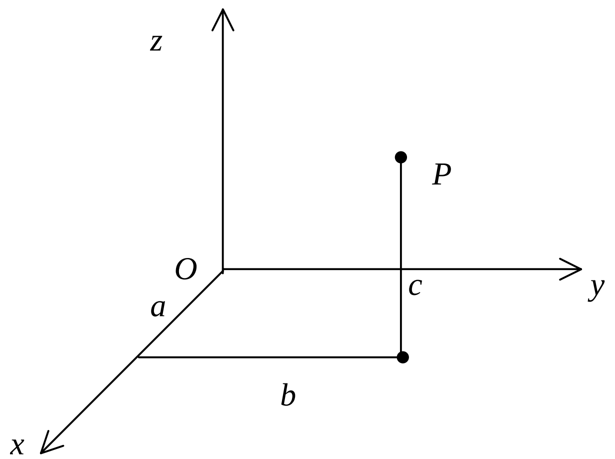
<u>**VECTORS**</u>

Three-dimensional coordinate systems

To locate a point in the three-dimensional space we require three numbers. In the three-dimensional space we have a fixed point *O*, referred to as the **origin**, three directed lines through the origin that are referred to as the **co-ordinate axes** (***x*-axis, *y*-axis and *z*-axis**). These three lines are mutually perpendicular. The three co-ordinate axes determine the **coordinate planes**. The *xy*-plane is the plane that contains the *x*- and *y*-axes. Similarly the *xz*-plane and the *yz*-plane are defined.

If *P* is a point in space, let *a* be the directed distance (this is the perpendicular distance) from *P* to the *yz*-plane. Similarly let *b* and *c* be the distances from *P* to the *xz*-plane and xy-plane respectively. We represent the point *P* by the ordered triple (*a*, *b*, *c*).



Distance: The **distance** $|PQ|$ between the point $P(a, b, c)$ and $Q(a_1, b_1, c_1)$ is given by
$|PQ| = \sqrt{(a_1 - a)^2 + (b_1 - b)^2 + (c_1 - c)^2}$

Equation of a Sphere: An equation of a sphere with centre $C(h, k, l)$ and radius r is given by $(x - h)^2 + (y - k)^2 + (z - l)^2 = r^2$.

VECTORS

A **vector** is a quantity that has both **magnitude** and **direction**. E.g.: wind movement described by speed and direction, say, 20 kph north east; force; displacement.

A **scalar** is a quantity described using just the magnitude. In this course a real number is referred to as a scalar**.**

Notation: We denote vectors using boldface lower case type such as **a**, **v**, **w** etc.

Vectors are represented geometrically by **arrows** in 2-space or 3-space. The direction of the arrow specifies the direction of the vector and the length of the arrow describes the magnitude of the vector.

The tail of the arrow is the **initial point** of the vector and the tip of the arrow is the **terminal point** of the vector.

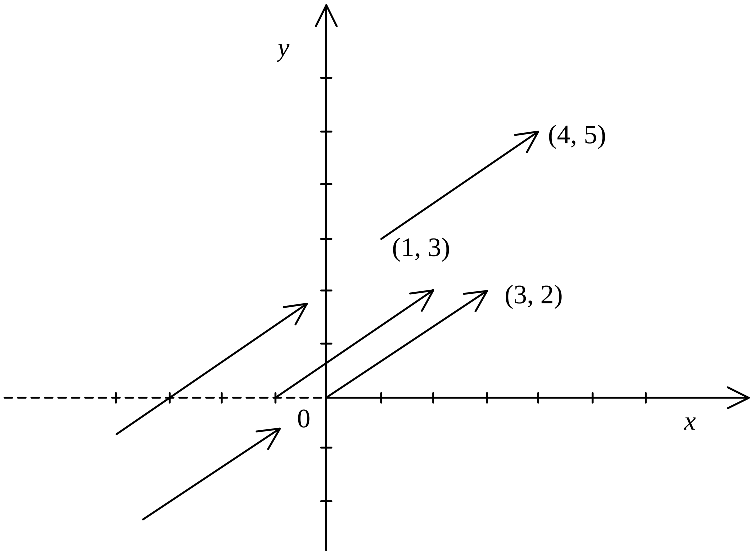


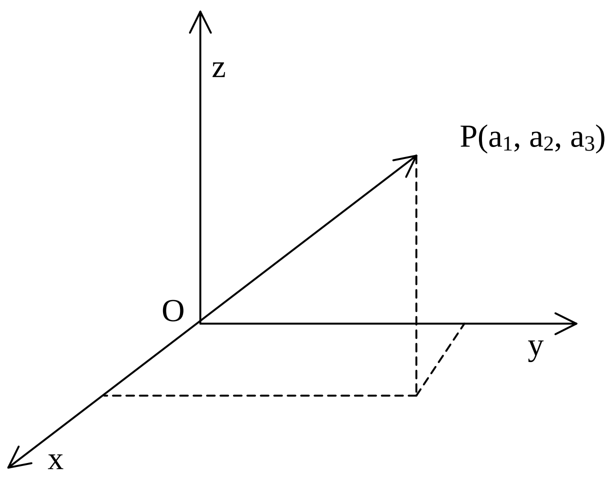
Figure 1

All the vectors represented by arrows in Figure 1 are **equivalent** since they have the same length and they point in the same direction (different positions).

The initial point of a vector can be moved to any convenient point A by an appropriate translation. If *A* is the initial point and *B* is the terminal point of **v**, then we write **v** = $\overrightarrow{AB}$. If the initial and terminal points of a vector coincide, then we have the **zero vector** denoted **0**.

ANALYTICAL REPRESENTATION

Fixed point O (origin). Three directed lines through O, **mutually perpendicular**: The coordinate axes x – axis, y – axis, z – axis. Place the initial point of a vector **a** at the origin O. Suppose that the terminal point of **a** has coordinates (a_1, a_2, a_3). The coordinates of the terminal point are referred to as the **components** of the vector **a.**



The particular representation of the vector $\overrightarrow{OP} = \langle a_1, a_2, a_3 \rangle$ = **a** from the origin to the point P is referred to as the **position vector** of the point P.

- Given points A(x_1, y_1, z_1) and B(x_2, y_2, z_2) the vector **a** with representation $\overrightarrow{AB}$ is $\langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle$. $\therefore a_1 = x_2 - x_1, a_2 = y_2 - y_1$, and $a_3 = z_2 - z_1$.

- The **length** of a vector **a** = $\langle a_1, a_2, a_3 \rangle$ is $|\mathbf{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$.

- The vector **0** = (0, 0, 0) has length 0. This is the only vector with length 0. This has no specific direction.

Example:

(1) Find the components of the vector with initial point P and terminal point Q.

(a) P(4, 8) and Q(3, 7) (b) P(3, -7, 2) and Q(-2, 5, -4)

Solution: (a) $\overrightarrow{PQ} = \langle 3-4, 7-8 \rangle = \langle -1, -1 \rangle$

(b) $\overrightarrow{PQ} = \langle -2-3, 5-(-7), -4-2 \rangle = \langle -5, 12, -6 \rangle$.

(2) Find a non zero vector **u** with initial point P(-1, 3, -5) such that
(a) u has the same direction as **v** = $\langle 6,7,-3 \rangle$ **(b) u** is oppositely directed to **v** = $\langle 6,7,-3 \rangle$

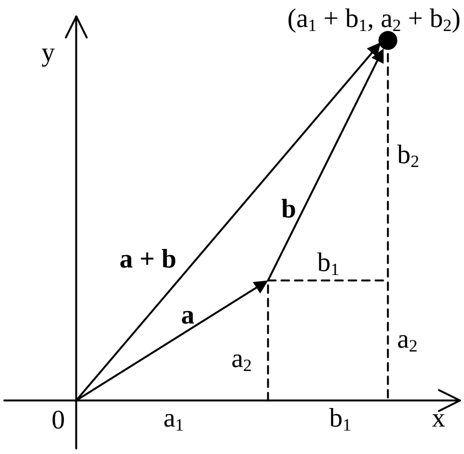
Definition:

A **two-dimensional vector** is an ordered pair **a** = $\langle a_1, a_2 \rangle$ of real numbers a_1 and a_2. A **three-dimensional vector** is an ordered triple **a** = $\langle a_1, a_2, a_3 \rangle$ of real numbers a_1, a_2 and a_3.

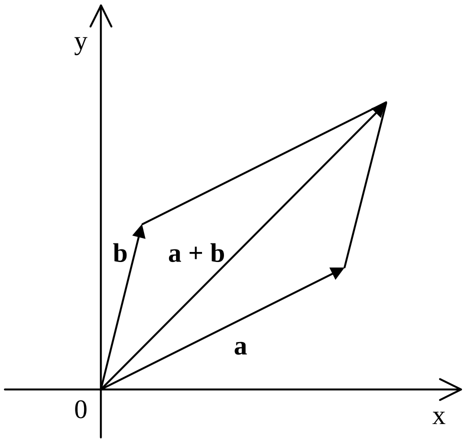
Vector Addition

If **a** = $\langle a_1, a_2 \rangle$ and **b** = $\langle b_1, b_2 \rangle$, then **a** + **b** = $\langle a_1 + b_1, a_2 + b_2 \rangle$. Similarly for three-dimensional vectors, $\langle a_1, a_2, a_3 \rangle + \langle b_1, b_2, b_3 \rangle = \langle a_1 + b_1, a_2 + b_2, a_3 + b_3 \rangle$.

- The **addition** of vectors is illustrated in the following figure. Geometrically, position the vectors **a** and **b** (without changing magnitudes or directions) so that the initial point of the vector **b** coincides with the terminal point of the vector **a**. The sum of the vectors **a** and **b** denoted **a+b** is the vector whose initial point coincides with the initial point of **a** and the terminal point coincides with the terminal point of **b.** This definition of addition of vectors is sometimes referred to as the **triangle law.**
- The sum of vectors **a** and **b** is also sometimes expressed through the so called **parallelogram law:** We draw the vectors **a** and **b** so that their initial points coincide. Now we can complete the parallelogram. The diagonal of the parallelogram that passes through the initial point of **a** (also the initial point of **b)** is the vector **a + b** .



Triangle Law



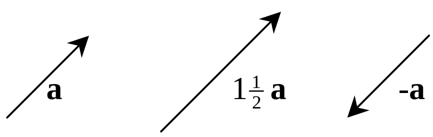
Parallelogram Law

Scalar Multiplication

If c is a scalar (real number) and **a** = $\langle a_1, a_2 \rangle$ is a vector , then the vector c**a** = $\langle ca_1, ca_2 \rangle$. Similarly for three-dimensional vectors, $c\langle a_1, a_2, a_3 \rangle = \langle ca_1, ca_2, ca_3 \rangle$

$$
\begin{aligned}
|c\mathbf{a}| &= |\langle ca_1, ca_2 \rangle| \\
&= \sqrt{(ca_1)^2 + (ca_2)^2} \\
&= \sqrt{c^2(a_1^2 + a_2^2)} \\
&= \sqrt{c^2}\sqrt{a_1^2 + a_2^2} \\
&= |c||\mathbf{a}| \text{ (Note that } |c| \text{ is the absolute value of the scalar c.)}
\end{aligned}
$$

- Length of c**a** = |c| × length of **a**.
- If c > 0 **a,** then and c**a** have the **same direction** and if c < 0, they have **opposite directions.**



Note: -**a** = (-1)**a**

Difference:

a – b = **a** + (-**b**) $= \langle a_1 - b_1, a_2 - b_2 \rangle$, where **a** $= \langle a_1, a_2 \rangle$; **b** $= \langle b_1, b_2 \rangle$

PROPERTIES OF VECTORS

Assume that **a**, **b**, **c** are vectors and x and y are scalars. We have the following properties:

1. **a** + **b** = **b** + **a** (commutative law)
2. **a** + (**b** + **c**) = (**a** + **b**) + **c** (associative law)
3. **a** + **0** = **a**
4. **a** + (-**a**) = **0**
5. x(**a** + **b**) = x**a** + x**b**
6. (x + y)**a** = x**a** + y**a**
7. (x.y)**a** = x(y**a**)
8. 1**a** = **a**

Def: A **unit vector** is a vector whose length is 1.

Question: How do you find a vector whose direction is the same as that of **u** and whose length is equal to 5?

Notes

- If **a** ≠ 0, the unit vector that has the same direction as **a** is $\mathbf{u} = \frac{\mathbf{a}}{|\mathbf{a}|}$.

Exercise: Verify that |**u**| = 1

- Two non-zero vectors **a** and **b** are **parallel** if each is a scalar multiple of the other That is, **a** = λ . **b** (λ: scalar).

Result 1: Two non-zero vectors **a** and **b** are parallel iff there are non-zero scalars λ and μ such that λ**a** + μ**b** = 0

Proof: Suppose **a** and **b** are parallel vectors.

⇒ There is a scalar s (≠ 0) such that **a** = s**b**
⇒ **a** – s**b** = **0**

Now choose λ = 1 and μ = -s. Thus we have the equation λ**a** + μ**b** = 0

Conversely suppose there are non-zero scalars λ and μ such that λ**a** + μ**b** = **0**

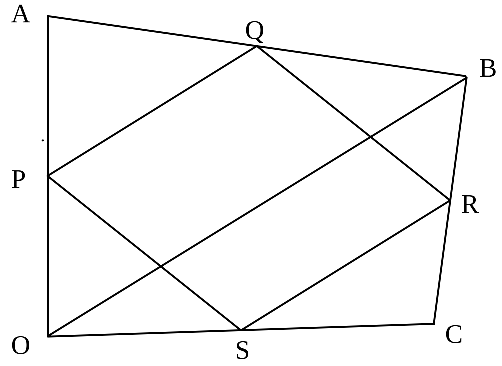
⇒ λ**a** = -μ**b**
⇒ $\mathbf{a} = \frac{-\mu}{\lambda}.\mathbf{b}$

Thus **a** and **b** are parallel vectors.

Corollary: If **a** and **b** are not parallel and λ**a** + μ**b** = **0** then λ = 0 and μ = 0.

USE OF VECTORS IN GEOMETRY

EXAMPLE: Prove that the midpoints of the sides of a quadrilateral form a parallelogram.



PROOF: Let OABC be a quadrilateral. Suppose P, Q, R and S are the midpoints of OA, AB, BC, and CO respectively.

It suffices to show that $\vec{PQ} = \vec{SR}$. (this would imply that these two sides are parallel and equal.) Since P is the mid-point of OA, $\vec{PA} = \frac{1}{2}\vec{OA}$. Similarly $\vec{AQ} = \frac{1}{2}\vec{AB}$.

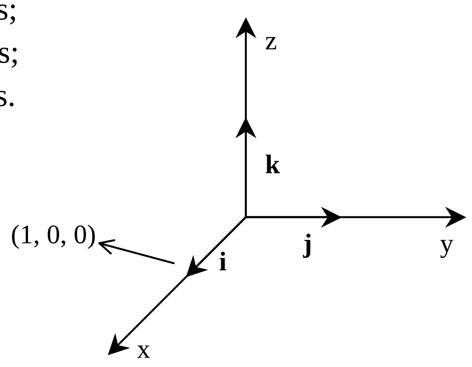
$$\begin{aligned}\therefore \vec{PQ} &= \vec{PA} + \vec{AQ}\\ &= \frac{1}{2}\vec{OA} + \frac{1}{2}\vec{AB}\\ &= \frac{1}{2}\left(\vec{OA} + \vec{AB}\right)\\ &= \frac{1}{2}\vec{OB}\end{aligned}$$

Similarly we can show that $\vec{SR} = \frac{1}{2}\vec{OB} \Rightarrow \vec{PQ} = \vec{SR}$ (Also check that $\vec{PS} = \vec{QR}$).

THREE DIMENSIONAL VECTORS

There are three special **unit vectors**.

i = $\langle 1, 0, 0 \rangle$ in the **positive** direction of x-axis;
j = $\langle 0, 1, 0 \rangle$ in the **positive** direction of y-axis;
k = $\langle 0, 0, 1 \rangle$ in the **positive** direction of z-axis.



Suppose **a** = $\langle a_1, a_2, a_3 \rangle$. Then

a = $\langle a_1,0,0\rangle + \langle 0,a_2,0\rangle + \langle 0,0,a_3\rangle$
= $a_1\langle 1,0,0\rangle + a_2\langle 0,1,0\rangle + a_3\langle 0,0,1\rangle$
= $a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$

Therefore, **a** can be expressed in terms of the unit vectors **i**, **j** and **k**.

For example, the vector $\langle 3, -5, 2\rangle = \langle 3, 0,0\rangle + \langle 0, -5, 0\rangle + \langle 0, 0, 2\rangle$
$= 3\langle 1, 0, 0)\rangle - 5\langle 0, 1, 0\rangle + 2\langle 0, 0, 1\rangle$
$= 3\mathbf{i} - 5\mathbf{j} + 2\mathbf{k}$.
The vectors **i**, **j** and **k** are referred to as **standard basis vectors.**

THE DOT PRODUCT

Consider two vectors **a** and **b** where **a** = $\langle a_1, a_2, a_3\rangle$ and **b** = $\langle b_1, b_2, b_3\rangle$. The **dot product** (or **scalar product**) of **a** and **b** is defined as: **a . b** = $a_1b_1 + a_2b_2 + a_3b_3$ (similar definition for two dimensional vectors).

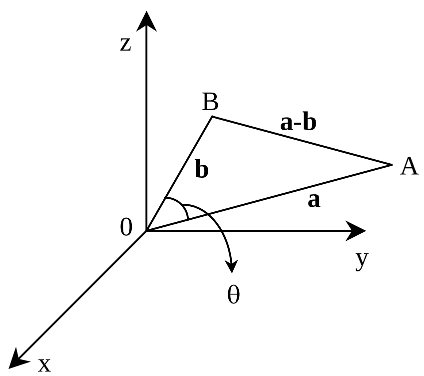
Properties: In the following, **a**, **b**, **c** are 3-dimensional vectors; t is a scalar.

1. **a . a** = $|\mathbf{a}|^2$
2. **a . b** = **b . a**
3. **a .** (**b** + **c**) = **a . b** + **a . c**
4. (t**a**) **. b** = t(**a . b**) = **a .** (t**b**)
5. **0 . a** = 0

For example, if **a** = $\langle 2, 6, -3\rangle$ and **b** = $\langle 8, -2, -1\rangle$ then **a . b** = $(2 \times 8) + (6 \times (-2)) + ((-3)\times(-1)) = 7$.

GEOMETRIC INTERPRETATION OF DOT PRODUCT

Consider the representations of **a** and **b** that start at the origin, let θ be the angle between OA and OB.



- Note that $0 \le \theta \le \pi$.

- For parallel vectors $\theta = 0$ or π

Theorem 2: $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}|\,|\mathbf{b}| \cos\theta$.

Proof: The law of cosines applied to the triangle OAB, gives

$$|AB|^2 = |OA|^2 + |OB|^2 - 2|OA|\,|OB| \cos\theta.$$

Since $|AB| = |\mathbf{a} - \mathbf{b}|$, $|OA| = |\mathbf{a}|$ and $|OB| = |\mathbf{b}|$, the above equation reduces to

$$|\mathbf{a} - \mathbf{b}|^2 = |\mathbf{a}|^2 + |\mathbf{b}|^2 - 2|\mathbf{a}|\,|\mathbf{b}| \cos\theta. \qquad (*)$$

Now using the properties of the dot product we have,

$$|\mathbf{a} - \mathbf{b}|^2 = (\mathbf{a} - \mathbf{b}) \cdot (\mathbf{a} - \mathbf{b})$$
$$= \mathbf{a} \cdot \mathbf{a} - \mathbf{a} \cdot \mathbf{b} - \mathbf{b} \cdot \mathbf{a} - \mathbf{b} \cdot \mathbf{b}$$
$$= |\mathbf{a}|^2 - 2\,\mathbf{a} \cdot \mathbf{b} + |\mathbf{b}|^2$$

Thus equation (*) gives

$$|\mathbf{a}|^2 - 2\,\mathbf{a} \cdot \mathbf{b} + |\mathbf{b}|^2 = |\mathbf{a}|^2 + |\mathbf{b}|^2 - 2|\mathbf{a}|\,|\mathbf{b}| \cos\theta$$

Hence $-2\,\mathbf{a} \cdot \mathbf{b} = -2\,|\mathbf{a}|\,|\mathbf{b}| \cos\theta$, and so $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}|\,|\mathbf{b}| \cos\theta$

Corollary: If θ is the angle between two non-zero vectors **a** and **b**, then

$$\cos\theta = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}||\mathbf{b}|}$$

Definition:

- Two non-zero vectors **a** and **b** are said to be **perpendicular** or **orthogonal** (to each other) if the angle θ between them is $\pi/2$.
- If **a** and **b** are orthogonal then $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}|\,|\mathbf{b}| \cos \pi/2 = 0$.
- Conversely if $\mathbf{a} \cdot \mathbf{b} = 0$ then $\cos\theta = \pi/2$.
- The zero vector **0** is perpendicular to all vectors.

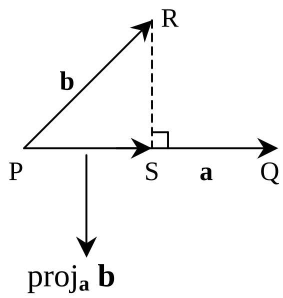
Result: **a** and **b** are orthogonal if and only if $\mathbf{a} \cdot \mathbf{b} = 0$.

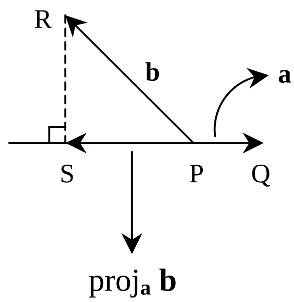
Remark: The scalar product $\mathbf{a} \cdot \mathbf{b}$ measures the extent to which **a** and **b** point in the same direction.

- If $0 \le \theta < \pi/2$ then $\cos\theta > 0$ and so $\mathbf{a} \cdot \mathbf{b} > 0$ (**a** and **b** point in the same general direction).
- If $\pi/2 < \theta \le \pi$ then $\cos\theta < 0$ and therefore $\mathbf{a} \cdot \mathbf{b} < 0$ (**a** and **b** point in generally opposite directions).
- In the extreme cases, i.e. when $\theta = 0$ or $\theta = \pi$ **a** and **b** point exactly in the same direction or exactly in the opposite direction.

APPLICATION OF DOT PRODUCT: PROJECTIONS

Consider vectors **a** and **b** with the same initial point P. Let $\mathbf{a} = \overrightarrow{PQ}$; $\mathbf{b} = \overrightarrow{PR}$. Let S be the foot of the perpendicular from R to the line containing $\overrightarrow{PQ}$.



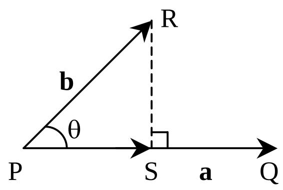


The vector with representation $\overrightarrow{PS}$ is called the **vector projection** of **b** onto **a** and is denoted by $\text{proj}_{\mathbf{a}}\, \mathbf{b}$.

The **scalar projection** of **b** onto **a** (component of b along a) is the magnitude of the vector projection, which is |b| cosθ, where θ is the angle between **a** and **b**.

Notation for scalar projection: $\text{comp}_{\mathbf{a}}\, \mathbf{b}$. Since $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}|\, |\mathbf{b}| \cos\theta$, it follows that the scalar projection

$$\text{comp}_{\mathbf{a}}\, \mathbf{b} = \frac{\mathbf{a}.\mathbf{b}}{|\mathbf{a}|} = \frac{\mathbf{a}}{|\mathbf{a}|}.\mathbf{b}$$



Note that the scalar projection is the dot product of **b** with the unit vector in the direction of **a**.

The vector projection of **b** onto **a**

= scalar projection × the unit vector in the direction of **a**

$$= \frac{\mathbf{a}.\mathbf{b}}{|\mathbf{a}|} \frac{\mathbf{a}}{|\mathbf{a}|}.$$

Thus $\text{proj}_{\mathbf{a}}\, \mathbf{b} = \dfrac{\mathbf{a}.\mathbf{b}}{|\mathbf{a}|^2} \mathbf{a}$.

Exercise: Show that the orthogonal projection of **b** denoted by $\mathbf{orth_a\, b = b - proj_a\, b}$ is orthogonal to the vector **a** .

Direction Angles and Direction Cosines

Let **a** = $\langle a_1, a_2, a_3 \rangle$ be a non-zero vector. The **direction angles** of **a** are the angles α, β and γ in the interval [0, π] that the vector **a** makes with the positive x-, y- and z-axes. cos α, cos β and cos γ are the **direction cosines** of **a**. Consider the dot product of **a** and **i**.

a $\bullet$ **i** = |**a**| |**i**| cos α. Clearly a $\bullet$ **i** = a_1. Thus

$a_1 = |\mathbf{a}| \bullet \cos\alpha$

$\therefore \cos\alpha = \dfrac{a_1}{|\mathbf{a}|}$. Similarly $\cos\beta = \dfrac{a_2}{|\mathbf{a}|}$; $\cos\gamma = \dfrac{a_3}{|\mathbf{a}|}$

Now, $\cos^2\alpha + \cos^2\beta + \cos^2\gamma$

$$= \frac{a_1^2}{|\mathbf{a}|^2} + \frac{a_2^2}{|\mathbf{a}|^2} + \frac{a_3^2}{|\mathbf{a}|^2}$$

$$= \frac{a_1^2 + a_2^2 + a_3^2}{|\mathbf{a}|^2}$$

$$= \frac{|\mathbf{a}|^2}{|\mathbf{a}|^2}$$

$$= 1$$

Hence we have $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$.

Also **a** = $\langle a_1, a_2, a_3 \rangle$

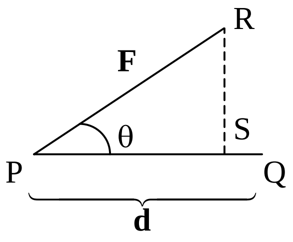
$= |\mathbf{a}| \langle \cos\alpha, \cos\beta, \cos\gamma \rangle$

i.e. $\dfrac{\mathbf{a}}{|\mathbf{a}|} = \langle \cos\alpha, \cos\beta, \cos\gamma \rangle$

Thus the direction cosines of a are the components of the unit vector in the direction of a.

WORK DONE BY A FORCE

Suppose that a **constant force F** acts on an object O which moves along a direction **other than** that of F.



F = $\overrightarrow{PR}$ (force)

d: displacement vector = $\overrightarrow{PQ}$

Work done W = component of **F** along **d** × distance moved
= |**F**| cos θ |**d**|
= **F** • **d** (dot product of **F** and **d**)

THE CROSS PRODUCT

Let **a**, **b** be vectors where **a** = $\langle a_1, a_2, a_3 \rangle$ and **b** = $\langle b_1, b_2, b_3 \rangle$.

The **cross product** **a** × **b** is the **vector** defined by

a × **b** = $\langle a_2 b_3 - a_3 b_2, a_3 b_1 - a_1 b_3, a_1 b_2 - a_2 b_1 \rangle$

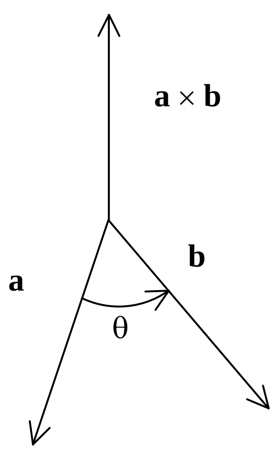
Example Let **a** = $\langle -3, 2, 2 \rangle$, **b** = $\langle 6, 3, 1 \rangle$

a × **b** = $\langle (2)(1) - (2)(3), (6)(2) - (-3)(1), (-3)(3) - (6)(2) \rangle$
= $\langle -4, 15, -21 \rangle$
or –4**i** + 15**j** – 21**k**

Result: For vectors **a** and **b, a** × **b** is orthogonal to **a** and **b** .
(**a** × **b**) • **a** = $a_1 (a_2 b_3 - a_3 b_2) + a_2 (a_3 b_1 - a_1 b_3) + a_3 (a_1 b_2 - a_2 b_1)$
= 0
Similarly, (**a** × **b**) • **b** = 0

Remark: **a** × **b** is defined only for three-dimensional vectors.

Direction of a × **b:** (right hand rule)



If the fingers of your right hand curl in the direction of a rotation (angle < 180°) from **a** to **b**, then the thumb points in the direction of **a** × **b**.

- **Theorem:** If θ is the angle between a and b ($0 \le \theta \le \pi$) then $|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}|\,|\mathbf{b}|\sin\theta$

- Two vectors **a** and **b** are parallel if and only if **a** × **b** = 0 (**a** and **b**: non zero)

- Proof of the statement $|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}|\,|\mathbf{b}|\sin\theta$

$$
\begin{aligned}
|\mathbf{a}\times\mathbf{b}|^2 &= (a_2b_3 - a_3b_1)^2 + (a_3b_1 - b_3a_1)^2 + (a_1b_2 - b_1a_2)^2 \\
&= (a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2) - (a_1b_1 + a_2b_2 + a_3b_3)^2 \\
&= |\mathbf{a}|^2|\mathbf{b}|^2 - (\mathbf{a}\cdot\mathbf{b})^2 \\
&= |\mathbf{a}|^2|\mathbf{b}|^2 - |\mathbf{a}|^2|\mathbf{b}|^2\cos^2\theta \\
&= |\mathbf{a}|^2|\mathbf{b}|^2[1 - \cos^2\theta] \\
&= |\mathbf{a}|^2|\mathbf{b}|^2\sin^2\theta \\
|\mathbf{a}\times\mathbf{b}| &= |\mathbf{a}||\mathbf{b}|\sin\theta \quad \left(\sqrt{\sin^2\theta} = \sin\theta \ge 0 \ \because, 0 \le \theta \le \pi\right)
\end{aligned}
$$

Theorem

Assume that **a, b, c** are vectors and s is a scalar.

1. **a** × **b** = **- b** × **a**
2. (s **a**) × **b** = s (**a** × **b**) = **a** × (s **b**)
3. **a** × (**b** + **c**) = (**a** × **b)** + (**a** × **c)**
4. (**a** + **b**) × **c** = (**a** × **c)** + (**b** × **c)**
5. **a** • (**b** × **c**) = (**a** × **b**) • **c**
6. **a** × (**b** × **c**) **=** (**a** • **c**) **b** – (**a** • **b**) **c**
7. **i** × **j** = **k** ; **j** × **k** = **i** ; **k** × **i** = **j**
8. The cross product is not associative ie **a** × (**b** × **c**) ≠ (**a** × **b**) × **c** (vector triple product)

Example

i × (**i** × **j**) = **i** × **k** = **-j** and (**i** × **i**) × **j** = **0** × **j** = **0**

Proof of 5:

Suppose **a** = $\langle a_1, a_2, a_3 \rangle$, **b** = $\langle b_1, b_2, b_3 \rangle$ and **c** = $\langle c_1, c_2, c_3 \rangle$.

Then **b** × **c** = $\langle b_2\, c_3 - b_3\, c_2, b_3\, c_1 - b_1\, c_3, b_1\, c_2 - b_2\, c_1 \rangle$.

Hence $\mathbf{a} \bullet (\mathbf{b} \times \mathbf{c}) = a_1(b_2\, c_3 - b_3\, c_2) + a_2\,(b_3\, c_1 - b_1\, c_3) + a_3\,(b_1\, c_2 - b_2\, c_1)$

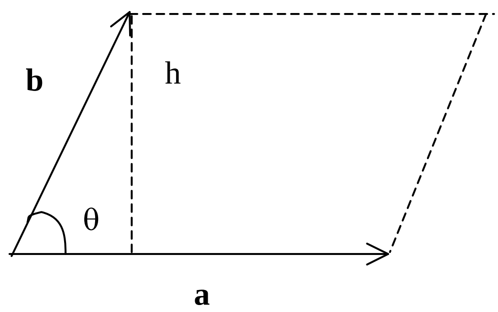
$= (a_2\, b_3 - a_3\, b_2)\, c_1 + (a_3\, b_1 - a_1\, b_3)\, c_2 + (a_1\, b_2 - a_2\, b_1)\, c_3$

$= (\mathbf{a} \times \mathbf{b}) \bullet \mathbf{c}$

Proof of 7: Recall **i** = $\langle 1,0,0 \rangle$ **j** = $\langle 0,1,0 \rangle$ and **k** = $\langle 0,0,1 \rangle$

$\mathbf{i} \times \mathbf{j} = \langle (0)(0) - (1)(0), (0)(0) - (1)(0), (1)(1) - (0)(0) \rangle = \langle 0,0,1 \rangle = \mathbf{k}.$

Geometric interpretation of a × b



$|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}|\, |\mathbf{b}| \sin\,\theta$

= (base) x (altitude)

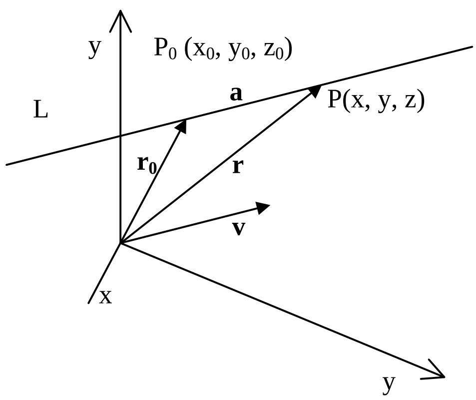
= **area of parallelogram determined by a and b.**

Definition: a • (b × c) is the **scalar triple product** of **a, b** and **c**.

Geometric significance of the scalar triple product: The volume of the parallelepiped determined by **a, b** and **c** is |**a • (b × c)**|.

Equation of a line in 3-space

A line L in the 3-dimensional space is determined if we know a point P_0 (x_0, y_0, z_0) and the direction of the line L. Let L be a line parallel to vector **v** and P(x, y, z) an arbitrary point on L. Assume that $\mathbf{r}_0$ and **r** are position vectors, of P_0 and P respectively.



Suppose **a** = $\overrightarrow{P_0P}$, then **r** = $\mathbf{r}_0$ + **a**   (Triangle Law)

Since **a** and **v** are parallel, **a** = t **v**, where t is a scalar.

$\therefore$ **r** = $\mathbf{r_0}$ + t **v**,   ------------ (1)

We refer to (1) as the **vector equation** of L. Each value of t (the parameter), gives the position vector of a point on L.

If the vector **v** = $\langle a, b, c\rangle$, then t**v** = $\langle ta, tb, tc\rangle$. We can write **r** = $\langle x, y, z\rangle$ and $\mathbf{r}_0$ = $\langle x_0, y_0, z_0\rangle$

The equation 1 is equivalent to $\langle x, y, z\rangle = \langle x_0 + ta, y_0 + tb, z_0 + tc\rangle$

$\Rightarrow$

$$\left.\begin{array}{l} x = x_0 + ta \\ y = y_0 + tb \\ z = z_0 + tc \end{array}\right\} \qquad \text{---------(2)}$$

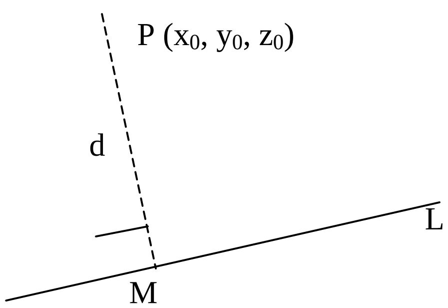
where t is a real number.

The equation 2 gives the **parametric equations** of line L through P_0 (x_0, y_0, z_0) parallel to vector **v** = $\langle a, b, c\rangle$. Each value of t gives a point P on L.

Eliminating t from (2), we have $\dfrac{x - x_0}{a} = \dfrac{y - y_0}{b} = \dfrac{z - z_0}{c}$ ----- (3)

These are the **symmetric equations of L** and **a, b, c** are the **direction numbers** of **L.**

Distance from a point to a line



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To find the distance d (shortest) from the point P to the line L:

Equation of line L: $x = x_1 + a_1 t$
$y = y_1 + a_2 t$

$z = z_1 + a_3 t$

Drop a perpendicular from P to the line L
Let M be the foot of the perpendicular line.
The coordinates of the point M are $(x_1 + a_1 t_1, y_1 + a_2 t_1, z_1 + a_3 t_1)$ for some t_1

$\overrightarrow{PM} = \langle x_1 + a_1 t_1 - x_0, y_1 + a_2\ t_1 - y_0, z_1 + a_3 t_1 - z_0 \rangle$

The direction of line L is $\langle a_1, a_2, a_3 \rangle = \mathbf{a}$

$\because \overrightarrow{PM} \bullet a = 0$, we have an equation for t_1. Now using the value of t_1 we can find the length $|\overrightarrow{PM}|$ of the vector $\overrightarrow{PM}$. Thus the shortest distance d from P to the line L is given by d = $|\overrightarrow{PM}|$.

Definitions: (1) Two lines which **do not intersect** and **are not parallel** are called **skew lines.**

(2) Points P, Q, R are **collinear** if they lie on a straight line i.e. $\overrightarrow{PQ}$ and $\overrightarrow{QR}$ are parallel vectors.

Eg: P (1, 0, 3), Q (0, 2, 4) and R (-2, 6, 6) are collinear since

$\overrightarrow{PQ} = \langle -1, 2, 1 \rangle$ and
$\overrightarrow{QR} = \langle -2, 4, 2 \rangle$ are parallel

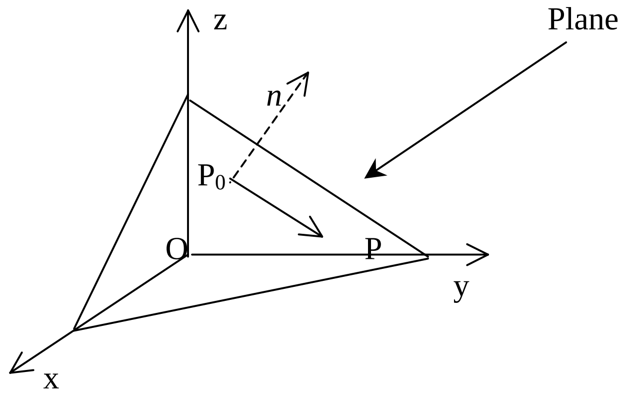
DISTANCE BETWEEN TWO SKEW LINES

Consider two skew lines L_1 and L_2. Let $\mathbf{s_1}$ and $\mathbf{s_2}$ be the directions of L_1 and L_2.
$\mathbf{n} = \mathbf{s}_1 \times \mathbf{s}_2$ is normal to both the lines. The distance between L_1 and L_2 is d, given by d = $|\overrightarrow{PQ} \bullet \hat{\mathbf{n}}|$ where P, Q are points on L_1 and L_2 and $\hat{\mathbf{n}}$ is the unit vector in the direction of **n**.

PLANES

Cartesian equation of a plane

A line in space is completely determined by a point and a direction. Similarly to completely describe a plane in space, we need a point and a point and a vector that is orthogonal to the plane. This orthogonal vector is called a **normal vector.** Let $P_0(x_0, y_0, z_0)$ be a fixed point on the plane and **n** = $\langle a, b, c\rangle$ a vector normal (orthogonal) to the plane. Let P (x, y, z) be an arbitrary point on the plane. Let $\mathbf{r_0}$ and **r** be the position



vectors of P_0 and P respectively. The vector $\mathbf{r} - \mathbf{r_0}$ is represented by $\overrightarrow{P_0P}$. The vector **n** is orthogonal to every vector in the plane , in particular, it is orthogonal to the vector $\overrightarrow{P_0P}$. Thus $\mathbf{n} \cdot (\mathbf{r} - \mathbf{r_0}) = 0$. That is $\mathbf{n} \cdot \mathbf{r} = \mathbf{n} \cdot \mathbf{r_0}$. This represents the **vector equation** of a plane . To obtain a Cartesian equation of a plane recall that $\overrightarrow{P_0P}$ = $\langle x - x_0,\ y - y_0,\ z - z_0\rangle$.

Since $\overrightarrow{P_0P}$ is perpendicular to **n** we have n • $\overrightarrow{P_0P}$ = 0

$\Leftrightarrow \langle a, b, c\rangle \bullet \langle x - x_0,\ y - y_0,\ z - z_0\rangle = 0.$ *Thus* $a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$

The above gives the **equation of the plane through P_0 and perpendicular to** **n =** $\langle a, b, c\rangle$ (**n** is **normal** to the plane)

The general form of the Cartesian equation of the plane is ax + by + cz = d

Definition

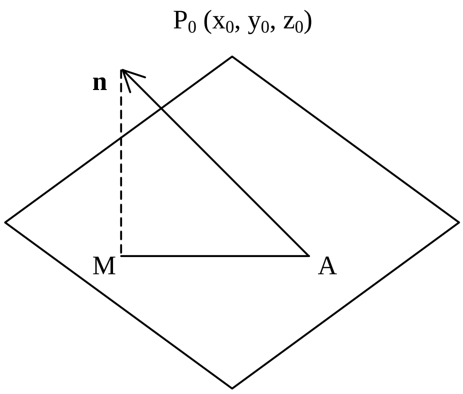
Two planes are **parallel** if their normal vectors are parallel.

Example: x + 2y – 3z = 4 : Plane 1
2x + 4y – 6z = 3 : Plane 2

Normal vectors for these planes are $\langle 1, 2, -3\rangle$ and $\langle 2, 4, -6\rangle$. These are parallel vectors (The components are multiples of each other). Hence the planes are parallel.

Note: If two planes are not parallel then they intersect in a line. The **angle between two planes** is the **acute angle** between their normal vectors.

Distance from a point to a plane



Plane: ax + by + cz = d
Point P_0 (x_0, y_0, z_0)
n: $\langle a, b, c\rangle$ normal to plane.

A: any point on the plane.

The distance from P_0 to the plane = $\left|\mathrm{Pr}\,oj\ \overrightarrow{AP}\ onto\ \mathbf{n}\right|$

$$= \left|\frac{\overrightarrow{AP}\bullet\mathbf{n}}{|\mathbf{n}|}\right|$$

Example: Point: P(1, -1, 0); Plane: x + y – z = 2 ; A: (2, 0, 0) is a point on the given plane.

Distance from P to the plane = $\left|\dfrac{\overrightarrow{AP}\bullet\langle 1,1,-1\rangle}{\sqrt{3}}\right| = \left|\dfrac{\langle -1, -1, 0\rangle\bullet\langle 1, 1, -1\rangle}{\sqrt{3}}\right| = \dfrac{2}{\sqrt{3}}$.