Difference Equations to **D**ifferential Equations

Section 5.9

Some Limit Calculations

In this section we will discuss the use of Taylor polynomials in computing certain types of limits. Although this material could have been treated directly after Section 5.2, we have saved it until now so as not to break into the development of Taylor series. To illustrate the ideas of this section, we begin with two examples, the first of which is already well-known to us.

Example Consider the problem of evaluating

$$\lim_{x \to 0} \frac{\sin(x)}{x}.$$

The reason this limit presents a problem is that, although the function in question is a quotient of two continuous functions, both the numerator and the denominator approach 0 as x approaches 0. Now from our work on Taylor polynomials we know that

$$\sin(x) = x + o(x),$$

so

$$\frac{\sin(x)}{x} = \frac{x + o(x)}{x} = 1 + \frac{o(x)}{x}.$$

But, by definition,

$$\lim_{x \to 0} \frac{o(x)}{x} = 0.$$

Thus

$$\lim_{x \to 0} \frac{\sin(x)}{x} = \lim_{x \to 0} \frac{x + o(x)}{x} = \lim_{x \to 0} \left(1 + \frac{o(x)}{x}\right) = 1.$$

Example The limit

$$\lim_{x \to 0} \frac{1 - \cos(x)}{x^2}$$

presents the same type of problem. Using the Taylor polynomial of order 2 for $\cos(x)$, we know that

$$\cos(x) = 1 - \frac{x^2}{2} + o(x^2).$$

Hence

$$\begin{aligned}
\lim_{x\to 0}\frac{1-\cos(x)}{x^2} &= \lim_{x\to 0}\frac{1-\left(1-\dfrac{x^2}{2}+o(x^2)\right)}{x^2}\\
&= \lim_{x\to 0}\frac{\dfrac{x^2}{2}-o(x^2)}{x^2}\\
&= \lim_{x\to 0}\left(\frac{1}{2}-\frac{o(x^2)}{x^2}\right)\\
&= \frac{1}{2}.
\end{aligned}$$

The point in both of these examples was to use the fact that if f is $n+1$ times continuously differentiable on an interval about the point c, then, as we saw in Section 5.2,

$$f(x)=f(c)+f'(c)(x-c)+\frac{f''(c)}{2!}(x-c)^2+\cdots+\frac{f^{(n)}(c)}{n!}+o((x-c)^n). \tag{5.9.1}$$

Hence if f and g are both $n+1$ times continuously differentiable on an interval about c, $f^{(n)}(c)\neq 0$, $f^{(k)}(c)=0$ for $k=0,1,2,\ldots,n-1$, and $g^{(k)}(c)=0$ for $k=0,1,2,\ldots,n-1$, then

$$f(x)=\frac{f^{(n)}(c)}{n!}(x-c)^n+o((x-c)^n) \tag{5.9.2}$$

and

$$g(x)=\frac{g^{(n)}(c)}{n!}(x-c)^n+o((x-c)^n). \tag{5.9.3}$$

Hence

$$\begin{aligned}
\lim_{x\to c}\frac{g(x)}{f(x)} &= \lim_{x\to c}\frac{\dfrac{g^{(n)}(c)}{n!}(x-c)^n+o((x-c)^n)}{\dfrac{f^{(n)}(c)}{n!}(x-c)^n+o((x-c)^n)}\\
&= \lim_{x\to c}\frac{\dfrac{g^{(n)}(c)}{n!}+\dfrac{o((x-c)^n)}{(x-c)^n}}{\dfrac{f^{(n)}(c)}{n!}+\dfrac{o((x-c)^n)}{(x-c)^n}}\\
&= \frac{\dfrac{g^{(n)}(c)}{n!}}{\dfrac{f^{(n)}(c)}{n!}}\\
&= \frac{g^{(n)}(c)}{f^{(n)}(c)}.
\end{aligned} \tag{5.9.4}$$

That is, under the specified conditions, the value of the limit is equal to the ratio of the nth derivatives of g and f evaluated at c. In addition, if it were the case that, for some $k < n$, $g^{(i)}(c) = 0$ for $i = 1, 2, \ldots, k-1$ and $g^{(k)}(c) \neq 0$, then we would have

$$\begin{aligned}\frac{g(x)}{f(x)} &= \frac{\dfrac{g^{(k)}(c)}{k!}(x-c)^k + o((x-c)^k)}{\dfrac{f^{(n)}(c)}{n!}(x-c)^n + o((x-c)^n)} \\ &= \frac{\dfrac{g^{(k)}(c)}{k!(x-c)^{n-k}} + \dfrac{o((x-c)^k)}{(x-c)^n}}{\dfrac{f^{(n)}(c)}{n!} + \dfrac{o((x-c)^n)}{(x-c)^n}} \\ &= \frac{\dfrac{1}{(x-c)^{n-k}}\left(\dfrac{g^{(k)}(c)}{k!} + \dfrac{o((x-c)^k)}{(x-c)^k}\right)}{\dfrac{f^{(n)}(c)}{n!} + \dfrac{o((x-c)^n}{(x-c)^n}}. \end{aligned} \tag{5.9.5}$$

Since the denominator of this last expression has a limit as x approaches c, but the numerator does not, it follows that in this case $\dfrac{g(x)}{f(x)}$ would not have a limit as x approaches c. That is, in this case $f(x)$ would approach 0 as $x \to c$ at a rate faster than $g(x)$, implying that the limit of the ratio would not exist.

In practice, we do not use the conclusions of the preceding paragraph, but rather apply the procedure outlined. That is, to evaluate

$$\lim_{x\to c} \frac{g(x)}{f(x)},$$

where both

$$\lim_{x\to c} g(x) = 0$$

and

$$\lim_{x\to c} f(x) = 0,$$

we replace both f and g by their respective Taylor polynomial expansions about c, expanded to the first nonzero term, and evaluate the limit as illustrated above.

Example To find

$$\lim_{x\to 0} \frac{\sin(x^2)}{x\tan(x)},$$

we note that

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

implies

$$\sin(x^2) = x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \frac{x^{14}}{7!} + \cdots,$$

so

$$\sin(x^2) = x^2 + o(x^2).$$

Moreover, using the first degree Taylor polynomial for $\tan(x)$ we have

$$\tan(x) = x + o(x).$$

Thus

$$\lim_{x\to 0} \frac{\sin(x^2)}{x\tan(x)} = \lim_{x\to 0} \frac{x^2 + o(x^2)}{x(x + o(x))} = \lim_{x\to 0} \frac{1 + \dfrac{o(x^2)}{x^2}}{1 + \dfrac{o(x)}{x}} = \frac{1+0}{1+0} = 1.$$

Example To evaluate

$$\lim_{x\to 0} \frac{x - \sin(x)}{x(1 - \cos(x))},$$

we first note that

$$\sin(x) = x - \frac{x^3}{3!} + o(x^3)$$

and

$$\cos(x) = 1 - \frac{x^2}{2} + o(x^2).$$

Then

$$x - \sin(x) = x - \left(x - \frac{x^3}{3!} + o(x^3)\right) = \frac{x^3}{3!} - o(x^3)$$

and

$$x(1 - \cos(x)) = x\left(1 - \left(1 - \frac{x^2}{2} + o(x^2)\right)\right) = \frac{x^3}{2} - xo(x^2).$$

Thus

$$\lim_{x\to 0} \frac{x - \sin(x)}{x(1 - \cos(x))} = \lim_{x\to 0} \frac{\dfrac{x^3}{3!} - o(x^3)}{\dfrac{x^3}{2} - xo(x^2)} = \lim_{x\to 0} \frac{\dfrac{1}{6} - \dfrac{o(x^3)}{x^3}}{\dfrac{1}{2} - \dfrac{o(x^2)}{x^2}} = \frac{\dfrac{1}{6} - 0}{\dfrac{1}{2} - 0} = \frac{1}{3}.$$

Example To evaluate

$$\lim_{x\to 0} \frac{\sin(x^2)}{x^4},$$

we first note that, as above,

$$\sin(x^2) = x^2 + o(x^2).$$

Hence

$$\lim_{x\to 0} \frac{\sin(x^2)}{x^4} = \lim_{x\to 0} \frac{x^2 + o(x^2)}{x^4} = \lim_{x\to 0} \left(\frac{1}{x^2} + \frac{o(x^2)}{x^4}\right) = \lim_{x\to 0} \frac{1}{x^2}\left(1 + \frac{o(x^2)}{x^2}\right) = \infty,$$

where the final equality follows after noting that

$$\lim_{x\to 0}\left(1+\frac{o(x^2)}{x^2}\right)=1,$$

while

$$\lim_{x\to 0}\frac{1}{x^2}=\infty.$$

The essence of (5.9.4) is also captured in the following statement, known as *l'Hôpital's rule.*

l'Hôpital's rule If f and g are twice continuously differentiable on an interval about the point c and both $g(c)=0$ and $f(c)=0$, then

$$\lim_{x\to c}\frac{g(x)}{f(x)}=\lim_{x\to c}\frac{g'(x)}{f'(x)}. \tag{5.9.6}$$

This is equivalent to our previous result, assuming the conditions specified at that time, because repeated applications of l'Hôpital's rule yield

$$\lim_{x\to c}\frac{g(x)}{f(x)}=\lim_{x\to c}\frac{g'(x)}{f'(x)}=\lim_{x\to c}\frac{g''(x)}{f''(x)}=\cdots=\lim_{x\to c}\frac{g^{(n)}(x)}{f^{(n)}(x)}=\frac{g^{(n)}(c)}{f^{(n)}(c)}, \tag{5.9.7}$$

which is (5.9.4). As before, if or some $k<n$, $g^{(i)}(c)=0$ for $i=0,1,2,\ldots,k-1$ and $g^{(k)}(c)\neq 0$, then

$$\lim_{x\to c}\frac{g^{(k)}(x)}{f^{(k)}(x)}$$

does not exist and $\dfrac{g(x)}{f(x)}$ does not have a limit as x approaches c.

Example We will illustrate l'Hôpital's rule first with another well-known limit. Namely,

$$\lim_{x\to 0}\frac{1-\cos(x)}{x}=\lim_{x\to 0}\frac{\frac{d}{dx}(1-\cos(x))}{\frac{d}{dx}x}=\lim_{x\to 0}\frac{\sin(x)}{1}=0.$$

Example As an illustration of how it may be necessary to apply l'Hôpital's rule more than once, we have

$$\begin{aligned}\lim_{x\to 0}\frac{\sin(x)-x}{x^3} &= \lim_{x\to 0}\frac{\dfrac{d}{dx}(\sin(x)-x)}{\dfrac{d}{dx}x^3}\\ &= \lim_{x\to 0}\frac{(\cos(x)-1)}{3x^2}\\ &= \lim_{x\to 0}\frac{\dfrac{d}{dx}(\cos(x)-1)}{\dfrac{d}{dx}3x^2}\\ &= \lim_{x\to 0}\frac{-\sin(x)}{6x}\\ &= -\frac{1}{6}\lim_{x\to 0}\frac{\sin(x)}{x}\\ &= -\frac{1}{6}.\end{aligned}$$

Note that this particular problem could have been done more quickly using the fact that

$$\sin(x)-x=-\frac{x^3}{6}+o(x^3).$$

Although we will not do so here, it is possible to demonstrate that l'Hôpital's rule is more widely applicable than what we have indicated so far. In particular, we may also apply l'Hôpital's rule to one-sided limits and to limits as x approaches ∞ or x approaches $-\infty$, provided, of course, that both $g(x)$ and $f(x)$ are approaching 0 and are twice continuously differentiable on the appropriate intervals. The following examples illustrate these applications.

Example Using l'Hôpital's rule, we have

$$\begin{aligned}\lim_{x\to\pi^+}\frac{\sin(x)}{\sqrt{x-\pi}} &= \lim_{x\to\pi^+}\frac{\dfrac{d}{dx}\sin(x)}{\dfrac{d}{dx}\sqrt{x-\pi}}\\ &= \lim_{x\to\pi^+}\frac{\cos(x)}{\dfrac{1}{2\sqrt{x-\pi}}}\\ &= \lim_{x\to\pi^+}2\cos(x)\sqrt{x-\pi}\\ &= 0.\end{aligned}$$

Example Using l'Hôpital's rule, we have

$$\begin{aligned}
\lim_{x\to\infty} x\sin\left(\frac{1}{x}\right) &= \lim_{x\to\infty} \frac{\sin\left(\frac{1}{x}\right)}{\frac{1}{x}} \\
&= \lim_{x\to\infty} \frac{\frac{d}{dx}\sin\left(\frac{1}{x}\right)}{\frac{d}{dx}\frac{1}{x}} \\
&= \lim_{x\to\infty} \frac{-\frac{1}{x^2}\cos\left(\frac{1}{x}\right)}{-\frac{1}{x^2}} \\
&= \lim_{x\to\infty} \cos\left(\frac{1}{x}\right) \\
&= 1.
\end{aligned}$$

Notice we could have computed this limit by substituting $h = \frac{1}{x}$, thus obtaining

$$\lim_{x\to\infty} x\sin\left(\frac{1}{x}\right) = \lim_{h\to 0^+} \frac{\sin(h)}{h} = 1.$$

Finally, it is also possible to demonstrate that l'Hôpital's rule applies when both the numerator and the denominator are approaching ∞. That is, if f and g are twice continuously differentiable at c and both

$$\lim_{x\to c} f(x) = \infty$$

and

$$\lim_{x\to c} g(x) = \infty,$$

then

$$\lim_{x\to c} \frac{g(x)}{f(x)} = \lim_{x\to c} \frac{g'(x)}{f'(x)}. \tag{5.9.8}$$

As before, this also applies for one-sided limits and for limits as x approaches ∞ or $-\infty$. Moreover, one or both of $g(x)$ and $f(x)$ may be approaching $-\infty$.

Example Using l'Hôpital's rule,

$$\begin{aligned}
\lim_{x\to\infty} \frac{3x+1}{\sqrt{x^2+4}} &= \lim_{x\to\infty} \frac{\frac{d}{dx}(3x+1)}{\frac{d}{dx}\sqrt{x^2+4}} \\
&= \lim_{x\to\infty} \frac{3}{\frac{x}{\sqrt{x^2+4}}} \\
&= \lim_{x\to\infty} \frac{3\sqrt{x^2+4}}{x} \\
&= \lim_{x\to\infty} 3\sqrt{\frac{x^2+4}{x^2}} \\
&= \lim_{x\to\infty} 3\sqrt{1+\frac{4}{x^2}} \\
&= 3.
\end{aligned}$$

Of course, we could have also computed this limit by dividing both the numerator and denominator by x to obtain

$$\lim_{x\to\infty} \frac{3x+1}{\sqrt{x^2+4}} = \lim_{x\to\infty} \frac{3+\frac{1}{x}}{\frac{\sqrt{x^2+4}}{x}} = \lim_{x\to\infty} \frac{3+\frac{1}{x}}{\sqrt{\frac{x^2+4}{x^2}}} = \lim_{x\to\infty} \frac{3+\frac{1}{x}}{\sqrt{1+\frac{4}{x^2}}} = 3.$$

Problems

1. Use Taylor polynomials to find the following limits.

   (a) $\displaystyle\lim_{x\to 0} \frac{\sin(3x)}{x}$

   (b) $\displaystyle\lim_{t\to 0} \frac{t-\sin(t)}{t^2}$

   (c) $\displaystyle\lim_{x\to 0} \frac{\cos(x)-1+\frac{x^2}{2}}{x^4}$

   (d) $\displaystyle\lim_{x\to 0} \frac{\sin(x^2)}{\sin^2(x)}$

   (e) $\displaystyle\lim_{u\to 0} \frac{\tan(u)}{\sin(u)}$

   (f) $\displaystyle\lim_{t\to 0} \frac{\sin(t)-t}{t^3}$

   (g) $\displaystyle\lim_{y\to 0} \frac{\tan(3y)}{\tan(5y)}$

   (h) $\displaystyle\lim_{x\to 0} \frac{\tan(x^2)}{\sin(x^2)}$

   (i) $\displaystyle\lim_{x\to 0} \frac{\sqrt{1+x}-1}{3x}$

   (j) $\displaystyle\lim_{t\to 0} \frac{\sqrt{1+t}-1-\frac{t}{2}}{3t^2}$

2. Use l'Hôpital's rule to evaluate the following limits.

   (a) $\lim_{x\to 0} \dfrac{\sin(5x)}{3x}$

   (b) $\lim_{t\to 0} \dfrac{1-\cos(3t)}{t^2}$

   (c) $\lim_{x\to 0} \dfrac{1-\sec(x)}{x}$

   (d) $\lim_{t\to \frac{\pi}{4}} \dfrac{1-\tan(t)}{\cos(2t)}$

   (e) $\lim_{x\to 0^+} \dfrac{\sin(2x)}{\sqrt{x}}$

   (f) $\lim_{x\to 0} \dfrac{1-\cos(x)}{\sin(x)}$

   (g) $\lim_{x\to \infty} x\sin\left(\dfrac{1}{x^2}\right)$

   (h) $\lim_{x\to 0} \dfrac{3x^2}{\sin^2(x)}$

3. Evaluate the following limits using any method you prefer.

   (a) $\lim_{x\to \infty} \dfrac{3x^2-2x+1}{16x^2+2}$

   (b) $\lim_{x\to 0} \dfrac{\tan(x^2)}{\sin^2(x)}$

   (c) $\lim_{x\to 0} \dfrac{1-\dfrac{\sin(x)}{x}}{3x^2}$

   (d) $\lim_{x\to 0} \dfrac{(1+x)^{\frac{1}{3}}-1}{x}$

   (e) $\lim_{x\to 0} \dfrac{(1+x)^{\frac{1}{3}}-1-\dfrac{x}{3}}{x^2}$

   (f) $\lim_{t\to 0} \dfrac{1-\cos(t)}{t\sin(2t)}$

   (g) $\lim_{x\to 1} \dfrac{\sqrt{x}-1}{x-1}$

   (h) $\lim_{x\to \pi} \dfrac{\cos(x)+1}{x-\pi}$

   (i) $\lim_{t\to 0} \dfrac{\cos(t)-1+\dfrac{t^2}{2}}{t^2\sin(t^2)}$

   (j) $\lim_{u\to 0} \dfrac{\sin(u^2)-u^2}{u^4(1-\cos(u))}$

4. Let $g(x) = \begin{cases} x^2\sin\left(\dfrac{1}{x}\right), & \text{if } x\neq 0, \\ 0, & \text{if } x = 0. \end{cases}$

   (a) Show that $g'(0) = 0$, and hence that $g(x) = o(x)$.

   (b) Use the preceding result and the fact that $\tan(x) = x + o(x)$ to show that

   $$\lim_{x\to 0} \frac{x^2\sin\left(\dfrac{1}{x}\right)}{\tan(x)} = 0.$$

   (c) Letting $f(x) = \tan(x)$, show that

   $$\lim_{x\to 0} \frac{g(x)}{f(x)} \neq \lim_{x\to 0} \frac{g'(x)}{f'(x)}.$$

   Which condition in the statement of l'Hôpital's rule does not hold for this example?