Section 3.9

The Geometry of Graphs

In Section 2.1 we discussed the graph of a function $y = f(x)$ in terms of plotting points $(x, f(x))$ for many different values of x and connecting the resulting points with straight lines. This is a standard procedure when using a computer and, if the function is well behaved and sufficiently many points are plotted, will produce a reasonable picture of the graph. However, as we noted at that time, this method assumes that the behavior of the graph between any two successive points is approximated well by a straight line. With a sufficient number of points and a differentiable function, this assumption will be reasonable. Yet to understand a graph fully, it is important to have alternative techniques to verify the picture at least qualitatively. We have already developed several important aids for understanding the shape of a graph, including techniques for determining the location of local extreme values and techniques for finding intervals where the function is increasing and intervals where it is decreasing. In this section we will use this information, along with additional information contained in the second derivative, to piece together a picture of the graph of a given function.

To see the importance of the second derivative, consider the graphs of $f(x) = x^2$ and $g(x) = \sqrt{x}$ on the interval $(0, \infty)$. Now

$$f'(x) = 2x$$
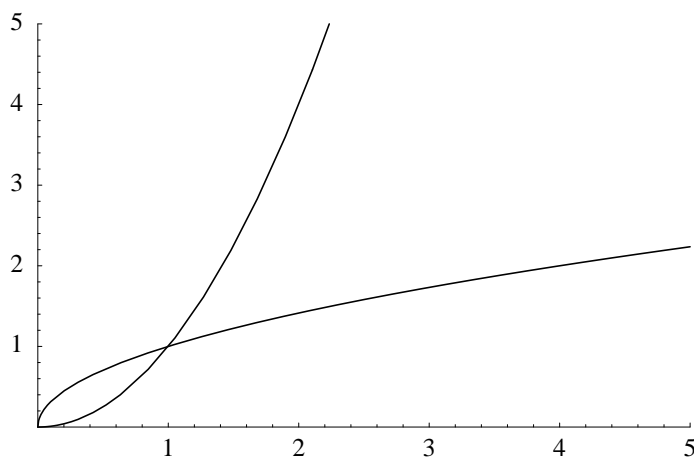
and

$$g'(x) = \frac{1}{2\sqrt{x}},$$
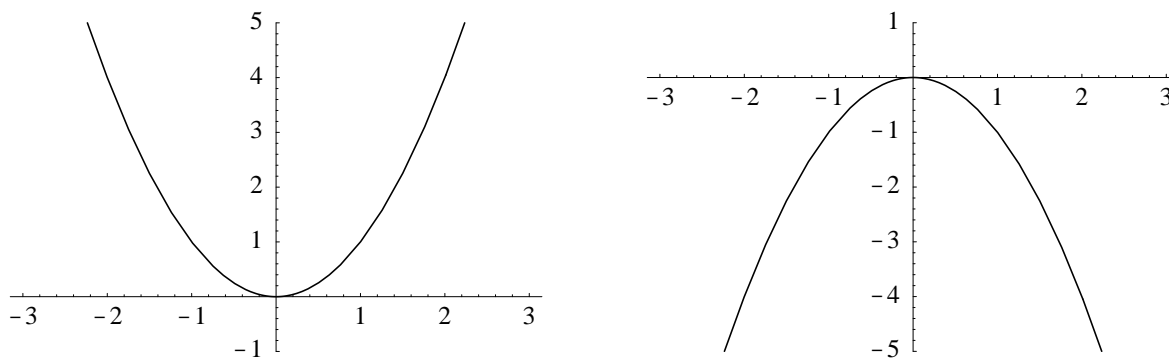

Figure 3.9.1 Graphs of $y = x^2$ and $y = \sqrt{x}$

Figure 3.9.2 Graphs of $y = x^2$ and $y = -x^2$ on $(-\infty, \infty)$

so $f'(x) > 0$ and $g'(x) > 0$ for all x in $(0, \infty)$. Thus f and g are both increasing on $(0, \infty)$. However, the graphs of f and g, as shown in Figure 3.9.1, are dramatically different. The graph of f is not only increasing, but is becoming steeper and steeper as x increases, whereas the graph of g is increasing, but flattening out as x increases. In other words, f' is itself an increasing function, causing the rate of growth of the function to increase with x, while g' is a decreasing function, resulting in a decrease in the rate of growth of g and a flattening out of the graph. In the terminology of the next definition, we say that the graph of f is concave up on $(0, \infty)$ and the graph of g is concave down on $(0, \infty)$.

Definition   Suppose f is differentiable on the open interval (a, b). If f' is an increasing function on (a, b), then we say the graph of f is *concave up* on (a, b). If f' is a decreasing function on (a, b), then we say the graph of f is *concave down* on (a, b).

Of course, to check for the intervals where f' is increasing and the intervals where f' is decreasing, we consider where f'', the derivative of f', is positive and where it is negative.

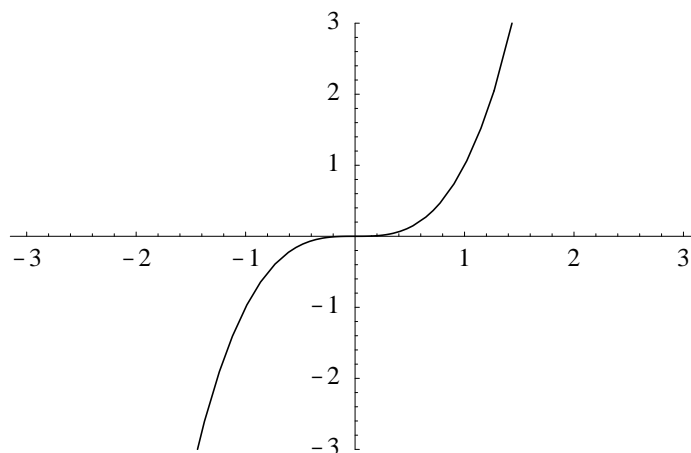**Proposition**   Suppose f is twice differentiable on the interval (a, b). If $f''(x) > 0$ for all x in (a, b), then the graph of f is concave up on (a, b); if $f''(x) < 0$ for all x in (a, b), then the graph of f is concave down on (a, b).

Example   Two basic examples to keep in mind are $f(x) = x^2$ and $g(x) = -x^2$. Since $f''(x) = 2 > 0$ and $g''(x) = -2 < 0$ for all values of x, the graph of f is concave up on $(-\infty, \infty)$ and the graph of g is concave down on $(-\infty, \infty)$. See Figure 3.9.2.

Example   Consider $g(t) = t^3$. Then $g''(t) = 6t$, so $g''(t) < 0$ when $t < 0$ and $g''(t) > 0$ when $t > 0$. Hence the graph of g is concave down on $(-\infty, 0)$ and concave up on $(0, \infty)$. Notice in Figure 3.9.3 how, even though g is increasing on $(-\infty, \infty)$, the change in concavity at $(0, 0)$ changes the shape of the graph.

Definition   A point on the graph of a function f where the concavity changes from up to down or from down to up is called an *inflection point.*

Example   In our previous example, $(0, 0)$ is an inflection point for the graph of $g(t) = t^3$.
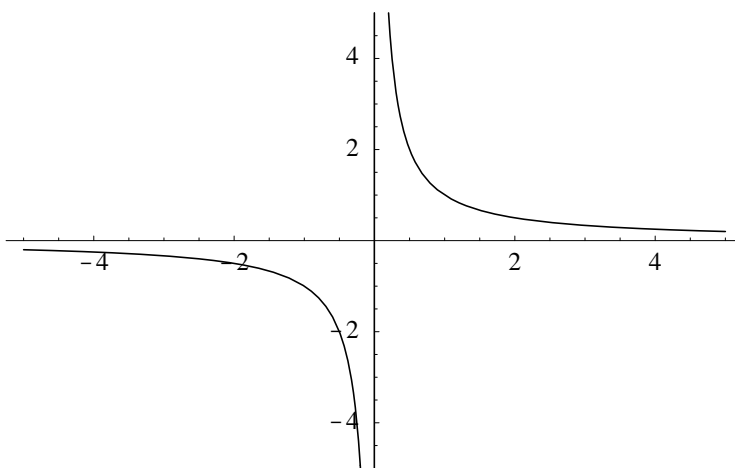
Figure 3.9.3 Graph of $g(t) = t^3$

Example Let $f(x) = \dfrac{1}{x}$. Then

$$f'(x) = -\frac{1}{x^2}$$

and

$$f''(x) = \frac{2}{x^3}.$$

Hence $f'(x) < 0$ on both $(-\infty, 0)$ and $(0, \infty)$, while $f''(x) < 0$ when $x < 0$ and $f''(x) > 0$ when $x > 0$. Thus f is decreasing on both $(-\infty, 0)$ and $(0, \infty)$, but the fact that the graph is concave down on $(-\infty, 0)$ shows up in the way the steepness of the graph increases as x approaches 0 from the right, while the fact that the graph is concave up on $(0, \infty)$ shows up in the way the graph flattens out as x increases toward ∞. See Figure 3.9.4. Also note that, although the concavity of the graph of f changes, the graph does not have an inflection point since f is not defined at 0.

Figure 3.9.4 Graph of $f(x) = \dfrac{1}{x}$

Note that if $(c, f(c))$ is an inflection point on the graph of a function f, then either $f''(c) = 0$ or f'' is not defined at c. However, the converse does not hold. For example, if $f(x) = x^4$, then $f''(0) = 0$, even though $f''(x) = 12x^2$ is positive for all x in both $(-\infty, 0)$ and $(0, \infty)$.

From the foregoing, it is clear that f' and f'' provide enough information to obtain a good understanding of the shape of the graph of f. Specifically, to sketch the graph of f, we use the first derivative to find (1) intervals where f is increasing, (2) intervals where f is decreasing, and (3) locations of any local extreme values; we use the second derivative to find (1) intervals where the graph of f is concave up, (2) intervals where the graph f is concave down, and (3) any inflection points. Combining this information with a few values of the function, the location of any asymptotes, and information on the behavior of $f(x)$ as x goes to $-\infty$ and as x goes to ∞, we can piece together a qualitatively accurate picture of the graph of f.

Example   Consider $f(x) = 3x^2 - x^3 + 2$. Then

$$f'(x) = 6x - 3x^2 = 3x(2 - x),$$

so the critical points of f are 0 and 2. Since $f'(-1) = -9 < 0$, $f'(1) = 3 > 0$, and $f'(3) = -9 < 0$, f is decreasing on the intervals $(-\infty, 0)$ and $(2, \infty)$ and increasing on $(0, 2)$. Moreover, this shows that f has a local minimum of 2 at $x = 0$ and a local maximum of 6 at $x = 2$.

Next, we have

$$f''(x) = 6 - 6x = 6(1 - x),$$

so $f''(x) = 0$ when $x = 1$. Now $1 - x > 0$ when $x < 1$ and $1 - x < 0$ when $x > 1$, so $f''(x) > 0$ on $(-\infty, 1)$ and $f''(x) < 0$ on $(1, \infty)$. Hence the graph of f is concave up on $(\infty, 1)$ and concave down on $(1, \infty)$, and $(1, 4)$ is an inflection point.

Combining this information with the values $f(-1) = 6$, $f(3) = 2$,

$$\lim_{x \to -\infty} f(x) = \lim_{x \to -\infty} (3x^2 - x^3 + 2) = \lim_{x \to -\infty} x^3 \left(\frac{3}{x} - 1 + \frac{2}{x^3} \right) = \infty,$$

and

$$\lim_{x \to \infty} f(x) = \lim_{x \to \infty} (3x^2 - x^3 + 2) = \lim_{x \to \infty} x^3 \left(\frac{3}{x} - 1 + \frac{2}{x^3} \right) = -\infty,$$

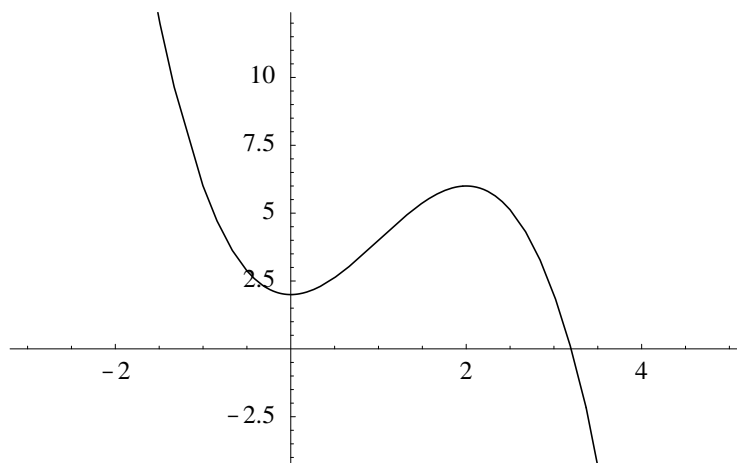we can easily draw a graph which, even though we are only plotting five points (the two local extreme values, the inflection point, and one point on each side of these points), captures the shape of the graph of f very well. See Figure 3.9.5.

Example   Consider $g(x) = 12x^5 + 15x^4 - 40x^3 - 10$. Then

$$g'(x) = 60x^4 + 60x^3 - 120x^2 = 60x^2(x^2 + x - 2) = 60x^2(x + 2)(x - 1),$$

implying that g has three critical points, namely, $x = -2$, $x = 0$, and $x = 1$. Now $60x^2 \geq 0$ for all values of x; $x + 2 < 0$ when $x < -2$ and $x + 2 > 0$ when $x > -2$; and $x - 1 < 0$

Figure 3.9.5 Graph of $f(x) = 3x^2 - x^3 + 2$

when $x < 1$ and $x - 1 > 0$ when $x > 1$. Thus $g'(x) > 0$ when $x < -2$ and when $x > 1$, and $g'(x) < 0$ when $-2 < x < 0$ and when $0 < x < 1$. So g is increasing on $(-\infty, -2)$ and $(1, \infty)$, and g is decreasing on $(-2, 0)$ and $(0, 1)$. In particular, g has a local maximum of 166 at $x = -2$ and a local minimum of -23 at $x = 1$. Although g has neither a local maximum nor a local minimum at the critical point 0, for drawing the graph of g it is important to note that the slope of the curve is 0 at $(0, -10)$.

Next,

$$g''(x) = 240x^3 + 180x^2 - 240x = 60x(4x^2 + 3x - 4),$$

so $g''(x) = 0$ when $x = 0$ and when $x^2 + 3x - 4 = 0$. Using the quadratic formula, the latter equation has solutions
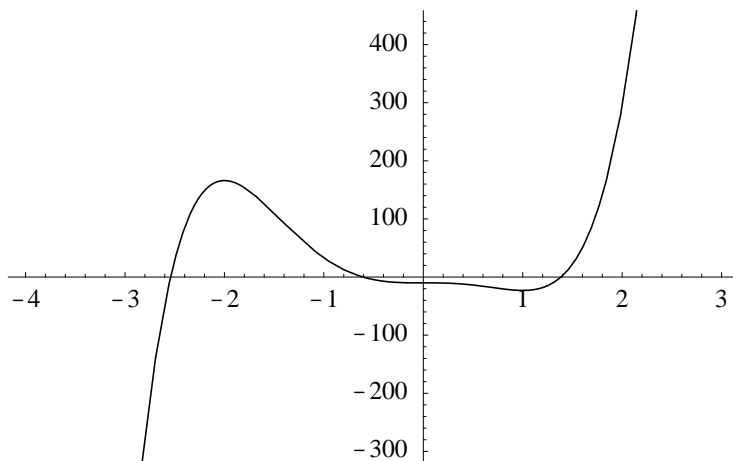
$$x = \frac{-3 - \sqrt{73}}{8} = -1.4430$$

and

$$x = \frac{-3 + \sqrt{73}}{8} = 0.6930,$$

rounding to four decimal places. Now $4x^2 + 3x - 4 < 0$ only when x is between the two roots -1.4430 and 0.6930. Since $60x < 0$ when $x < 0$ and $60x > 0$ when $x > 0$, we may conclude that $g''(x) < 0$ for $x < -1.4430$ and $0 < x < 0.6930$, and $g''(x) > 0$ for $-1.4430 < x < 0$ and $x > 0.6930$. Hence the graph of g is concave down on $(-\infty, -1.4430)$ and $(0, 0.6930)$ and concave up on $(-1.4430, 0)$ and $(0.6930, \infty)$. In particular, g has three inflection points: $(-1.4430, 100.1459)$, $(0, -10)$, and $(0.6930, -17.9349)$.

Adding to this information the values $g(-3) = -631$, $g(2) = 294$,

$$\lim_{x \to -\infty} g(x) = \lim_{x \to -\infty} (12x^5 + 15x^4 - 40x^3 - 10)$$

$$= \lim_{x \to -\infty} x^5 \left(12 + \frac{15}{x} - \frac{40}{x^2} - \frac{10}{x^5} \right)$$

$$= -\infty,$$

Figure 3.9.6 Graph of $g(x) = 12x^5 + 15x^4 - 40x^3 - 10$

and

$$\lim_{x \to \infty} g(x) = \lim_{x \to \infty} (12x^5 + 15x^4 - 40x^3 - 10)$$

$$= \lim_{x \to \infty} x^5 \left(12 + \frac{15}{x} - \frac{40}{x^2} - \frac{10}{x^5} \right)$$

$$= \infty,$$

we can now sketch the graph of g. See Figure 3.9.6.

Example   For our final example, consider

$$h(t) = \frac{t^2}{t^2 - 1}.$$

Then

$$h'(t) = \frac{(t^2 - 1)(2t) - (t^2)(2t)}{(t^2 - 1)^2} = -\frac{2t}{(t^2 - 1)^2},$$

so $h'(t) = 0$ when $2t = 0$. Thus h has one critical point, $t = 0$. However, we must also take into consideration the two points where h and h' are not defined, namely, $t = -1$ and $t = 1$. Now $(t^2 - 1)^2 \geq 0$ for all t, so the sign of h' is determined by the sign of $-2t$. Thus $h'(t) > 0$ when $t < -1$ and when $-1 < t < 0$, and $h'(t) < 0$ when $0 < t < 1$ and when $t > 1$. In other words, h is increasing on $(-\infty, -1)$ and $(-1, 0)$, and h is decreasing on $(0, 1)$ and $(1, \infty)$. From this we see that h has a local maximum of 0 at $t = 0$. For the second derivative, we have

$$h''(t) = \frac{(t^2 - 1)^2(-2) - (-2t)(2(t^2 - 1)(2t))}{(t^2 - 1)^4} = \frac{-2(t^2 - 1) + 8t^2}{(t^2 - 1)^3} = \frac{6t^2 + 2}{(t^2 - 1)^3}.$$

Since $6t^2 + 2 > 0$ for all values of t, it follows that $h''(t) \neq 0$ for all t. However, as with the first derivative, we need to consider the points $t = -1$ and $t = 1$ where h'' is not defined. Now $t^2 - 1 < 0$ only when $-1 < t < 1$, so $h''(t) < 0$ when $-1 < t < 1$ and $h''(t) > 0$ when
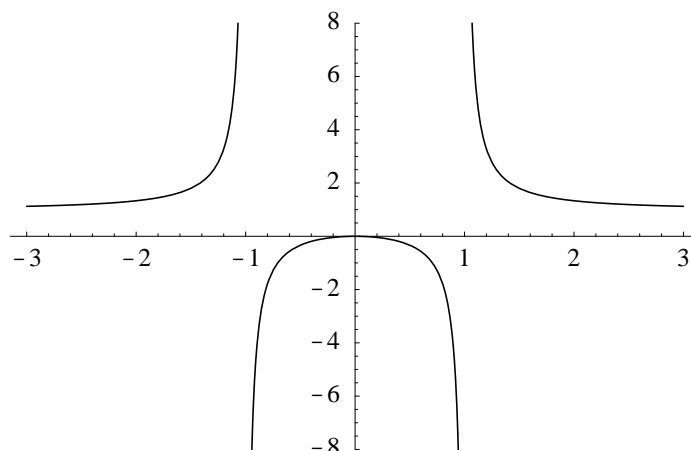
Figure 3.9.7 Graph of $h(t) = \dfrac{t^2}{t^2 - 1}$

$t < -1$ and when $t > 1$. Hence the graph of h is concave down on $(-1, 1)$ and concave up on $(-\infty, -1)$ and $(1, \infty)$. Note, however, that there are no points of inflection.

Since h is not defined at $t = -1$ and $t = 1$, we need to check for vertical asymptotes at these points. We have

$$\lim_{t \to -1^-} h(t) = \lim_{t \to -1^-} \frac{t^2}{t^2 - 1} = \infty,$$

$$\lim_{t \to -1^+} h(t) = \lim_{t \to -1^+} \frac{t^2}{t^2 - 1} = -\infty,$$

$$\lim_{t \to 1^-} h(t) = \lim_{t \to 1^-} \frac{t^2}{t^2 - 1} = -\infty,$$

and

$$\lim_{t \to 1^+} h(t) = \lim_{t \to 1^+} \frac{t^2}{t^2 - 1} = \infty,$$

showing that the graph of h has vertical asymptotes at $t = -1$ and $t = 1$. Finally,

$$\lim_{t \to -\infty} h(t) = \lim_{t \to -\infty} \frac{t^2}{t^2 - 1} = \lim_{t \to -\infty} \frac{1}{1 - \dfrac{1}{t^2}} = 1$$

and

$$\lim_{t \to \infty} h(t) = \lim_{t \to \infty} \frac{t^2}{t^2 - 1} = \lim_{t \to \infty} \frac{1}{1 - \dfrac{1}{t^2}} = 1$$

show that the graph of h has a horizontal asymptote at $y = 1$. With all of this geometric information, we may now draw the graph of h, as shown in Figure 3.9.7.

Problems

1. Discuss the geometry of the graphs of each of the following functions. That is, find the intervals where the function is increasing and where it is decreasing, find the intervals where the graph is concave up and where it is concave down, find all local extreme values and where they are located, find all inflection points, find any vertical or horizontal asymptotes, and use this information to sketch the graph.

   (a) $f(x) = x^2 - x$ (b) $g(t) = 3t^2 + 2t - 6$

   (c) $g(x) = x^3 + 3x^2$ (d) $f(t) = t^4 + 2t^2$

   (e) $f(x) = x^3 - 3x$ (f) $g(x) = 3x^5 - 5x^3$

   (g) $h(x) = x^5 - x^3$ (h) $f(x) = 3x^5 - 5x^4$

   (i) $g(z) = \dfrac{1}{z-1}$ (j) $y(t) = \dfrac{1}{t^2+1}$

   (k) $f(x) = x^4 - 2x^3$ (l) $h(t) = \dfrac{t}{1+t^2}$

   (m) $h(t) = \dfrac{t}{t^2-4}$ (n) $g(x) = \dfrac{x}{1+3x^2}$

   (o) $f(x) = \dfrac{x^2}{1+x^2}$ (p) $f(x) = \dfrac{1}{x^2-1}$

   (q) $x(t) = \dfrac{2t+1}{t-1}$ (r) $f(z) = \dfrac{z^2}{z^2-4}$

2. Suppose the function f has the following properties:

$$f(0) = 0$$
$$f'(x) > 0 \text{ for } x \text{ in } (-\infty, 2)$$
$$f'(x) < 0 \text{ for } x \text{ in } (2, \infty)$$
$$f''(x) < 0 \text{ for } x \text{ in } (-2, 6)$$
$$f''(x) > 0 \text{ for } x \text{ in } (-\infty, -2) \text{ and for } x \ (6, \infty)$$
$$\lim_{x \to -\infty} f(x) = -2$$
$$\lim_{x \to \infty} f(x) = 0$$

   Sketch the graph of a function satisfying these conditions.

3. Suppose $f(0) = 0$ and $f'(x) = x^2 - 1$.

   (a) Sketch what the graph of f must look like.

   (b) Graph f' on the same axes with f.

   (c) Is there more than one function f which satisfies these conditions?

4. Suppose $f(0) = 0$ and $f'(x) = x^3 + x^2 - 6x$.

   (a) Sketch what the graph of f must look like.

   (b) Graph f' on the same axes with f.

   (c) Is there more than one function f which satisfies these conditions?

5. Suppose $g(1) = 0$ and $g'(t) = \dfrac{1}{t}$.

   (a) Sketch what the graph of g must look like on $(0, \infty)$.

   (b) Graph g' on the same axes with g.

   (c) Is there more than one function g which satisfies these conditions?

6. Suppose $f(0) = 1$ and $f'(x) = f(x)$. What must the graph of f look like? Is this enough information to determine the graph of f?