Lecture 6: Limits

6.1 Definition of a limit

Idea: Given a function f, we say $\lim_{x \to a} f(x) = L$ (i.e., the limit of $f(x)$ as x approaches a is L) if we can make $f(x)$ arbitrarily close to L by choosing x sufficiently close to a (without having $x = a$).

Note: Distance from $f(x)$ to L is $|f(x) - L|$ and the distance from x to a is $|x - a|$.

Definition We say the *limit* of $f(x)$ as x approaches a is L, written

$$\lim_{x \to a} f(x) = L,$$

if for every $\epsilon > 0$ there exists a number $\delta > 0$ such that

$$|f(x) - L| < \epsilon \text{ whenever } 0 < |x - a| < \epsilon.$$

Example Suppose $f(x) = x^2$. Then $\lim_{x \to 2} f(x) = 4$.

Example Suppose

$$f(x) = \begin{cases} x^2, & \text{if } x \leq 0 \\ x + 1, & \text{if } x > 0. \end{cases}$$

Then, for example, $\lim_{x \to -1} f(x) = 1$ and $\lim_{x \to 1} f(x) = 2$. But $\lim_{x \to 0} f(x)$ does not exist. However, we could write

$$\lim_{x \to 0^-} f(x) = 0,$$

which we call a *left-hand limit*, and

$$\lim_{x \to 0^+} f(x) = 1,$$

which we call a *right-hand limit.*

Note: $\lim_{x \to a} f(x) = L$ if and only if both $\lim_{x \to a^-} f(x) = L$ and $\lim_{x \to a^+} f(x) = L$

6.2 Some more examples

Example Suppose

$$f(x) = \begin{cases} x - 1, & \text{if } x \leq 0 \\ x + 1, & \text{if } 0 < x < 2 \\ 5 - x, & \text{if } x \geq 2. \end{cases}$$

Then, for example,

$$\lim_{x\to 4} f(x) = \lim_{x\to 4}(5-x) = 1,$$

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-}(x-1) = -1,$$

and

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+}(x+1) = 1.$$

It follows from the latter two limits that $\lim_{x\to 0} f(x)$ does not exist. On the other hand,

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-}(x+1) = 3$$

and

$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+}(5-x) = 3,$$

from which it follows that $\lim_{x\to 2} f(x) = 3$.

Example Let $f(x) = \sin(\frac{1}{x})$. Then

$$f(\frac{1}{\pi}) = 0, f(\frac{1}{2\pi}) = 0, f(\frac{1}{3\pi}) = 0, \ldots$$

while

$$f(\frac{2}{\pi}) = 1, f(\frac{2}{5\pi}) = 1, f(\frac{2}{9\pi}) = 1, \ldots.$$

Hence $\lim_{x\to 0} \sin(\frac{1}{x})$ does not exist.

6.3 Unbounded limits and asymptotes

Example $\lim_{x\to 0} \frac{1}{x^2}$ does not exist, but we will write $\lim_{x\to 0} \frac{1}{x^2} = \infty$. Note that the line $x = 0$ is a vertical asymptote for the graph of $y = \frac{1}{x^2}$.

Example $\lim_{x\to 0^-} \frac{1}{x} = -\infty$ and $\lim_{x\to 0^+} \frac{1}{x} = \infty$, and once again the line $x = 0$ is a vertical asymptote for the graph of $y = \frac{1}{x}$.

In general, any one of the following indicates that the line $x = a$ is a vertical asymptote for the graph of the function f:

$$\lim_{x\to a^+} f(x) = \infty, \lim_{x\to a^+} f(x) = -\infty, \lim_{x\to a^-} f(x) = \infty, \lim_{x\to a^-} f(x) = -\infty.$$

Example Since $\lim_{x\to 2^-} \frac{4}{2-x} = \infty$ and $\lim_{x\to 2^+} \frac{4}{2-x} = -\infty$, the line $x = 2$ is a vertical asymptote for the graph of $y = \frac{4}{2-x}$.