Lecture 8: Definition of Limit

8.1 The definition of limit revisited

Recall: $\lim_{x \to a} f(x) = L$ means that for every $\epsilon > 0$ there exists a $\delta > 0$ such that

$$|f(x) - L| < \epsilon \text{ whenever } 0 < |x - a| < \delta.$$

Example From our previous work we know that $\lim_{x \to 2}(4x - 1) = 7$. So we might ask, "How close to 2 does x have to be in order that $f(x) = 4x - 1$ is within 0.01 of 7?" That is, we want

$$|f(x) - 7| = |4x - 1 - 7| = |4x - 8| < 0.01.$$

Now $|4x - 8| = 4|x - 2|$, so

$$|4x - 8| < 0.01 \iff 4|x - 2| < 0.01 \iff |x - 2| < \frac{0.01}{4} = 0.0025.$$

That is, $|(4x - 1) - 7| < 0.01$ whenever $0 < |x - 2| < 0.0025$.

More generally, for any $\epsilon > 0$,

$$|4x - 8| < \epsilon \iff 4|x - 2| < \epsilon \iff |x - 2| < \frac{\epsilon}{4}.$$

Hence if we take $\delta = \frac{\epsilon}{4}$, then $|(4x - 1) - 7| < \epsilon$ whenever $0 < |x - 2| < \delta$, thus verifying, by the definition, that $\lim_{x \to 2}(4x - 1) = 7$.

Example We will verify that $\lim_{x \to 12}(\frac{x}{4} - 4) = -1$. That is, given $\epsilon > 0$, we want to find $\delta > 0$ such that $|(\frac{x}{4} - 4) + 1| < \epsilon$ whenever $0 < |x - 12| < \delta$. Now

$$|(\frac{x}{4} - 4) + 1| = |\frac{x}{4} - 3| = \frac{1}{4}|x - 12|,$$

so

$$|(\frac{x}{4} - 4) + 1| < \epsilon \iff |x - 12| < 4\epsilon.$$

Hence if we take $\delta = 4\epsilon$, then $|(\frac{x}{4} - 4) + 1| < \epsilon$ whenever $0 < |x - 12| < \delta$, showing that $\lim_{x \to 12}(\frac{x}{4} - 4) = -1$.