Difference Equations to Differential Equations

Section 8.3

First Order Linear Differential Equations

We will now consider closed form solutions for another important class of differential equations. A differential equation

$$\dot{x} = f(x,t)$$

with $x(t_0) = x_0$ is called a *linear equation* if

$$f(x,t) = p(t)x + q(t) \tag{8.3.1}$$

for some functions p and q which depend only on t. We will assume that both p and q are continuous functions. Note that under certain circumstances, such as $q(t) = 0$ for all t, a linear equation is also separable. The solution of such equations is based on the following observation: If we let

$$P(t) = \int_{t_0}^{t} p(s)ds, \tag{8.3.2}$$

then

$$\frac{d}{dt}\left(xe^{-P(t)}\right) = -xp(t)e^{-P(t)} + \dot{x}e^{-P(t)} = e^{-P(t)}(\dot{x} - p(t)x). \tag{8.3.3}$$

Now we want $\dot{x} = p(t)x + q(t)$, that is, $\dot{x} - p(t)x = q(t)$, so we are looking for a function x such that

$$\frac{d}{dt}\left(xe^{-P(t)}\right) = q(t)e^{-P(t)}. \tag{8.3.4}$$

Integrating (8.3.4) from t_0 to t (using u for our variable of integration), we have

$$\int_{t_0}^{t} \frac{d}{du}\left(x(u)e^{-P(u)}\right)du = \int_{t_0}^{t} q(u)e^{-P(u)}du. \tag{8.3.5}$$

Now

$$\begin{aligned}\int_{t_0}^{t} \frac{d}{du}\left(x(u)e^{-P(u)}\right)du &= x(u)e^{-P(u)}\Big|_{t_0}^{t} \\ &= x(t)e^{-P(t)} - x(t_0)e^{-P(t_0)} \\ &= x(t)e^{-P(t)} - x_0\end{aligned} \tag{8.3.6}$$

since $P(t_0) = 0$ and $x(t_0) = x_0$. Hence we want

$$x(t)e^{-P(t)} - x_0 = \int_{t_0}^{t} q(u)e^{-P(u)}du. \tag{8.3.7}$$

Solving (8.3.7) fot $x(t)$, we have

$$x(t) = e^{P(t)} \left(\int_{t_0}^{t} q(u) e^{-P(u)} du + x_0 \right). \tag{8.3.8}$$

Similar to our situation with separable equations, (8.3.8) provides a closed form solution to our equation only if the requisite integrals may be computed in closed form. If not, numerical techniques will be necessary.

Linear equations If p and q are continuous, x satisfies the differential equation

$$\dot{x} = p(t)x + q(t) \tag{8.3.9}$$

with $x(t_0) = x_0$, and

$$P(t) = \int_{t_0}^{t} p(s) ds, \tag{8.3.10}$$

then

$$x(t) = e^{P(t)} \left(\int_{t_0}^{t} q(u) e^{-P(u)} du + x_0 \right). \tag{8.3.11}$$

Example Consider the equation

$$\dot{x} = \frac{x}{t} + 4t$$

with $x(1) = 5$. This is a linear equation with, in the notation used above,

$$p(t) = \frac{1}{t}$$

and

$$q(t) = 4t.$$

Then

$$P(t) = \int_{1}^{t} \frac{1}{s}\, ds = \log(s)\Big|_{1}^{t} = \log(t),$$

where, in order for the integral to exist, we have restricted t to be positive. Thus, using (8.3.11),

$$\begin{aligned} x &= e^{\log(t)} \left(\int_{1}^{t} 4u e^{-\log(u)} du + 5 \right) \\ &= t \left(\int_{1}^{t} \frac{4u}{u}\, du + 5 \right) \\ &= t \left(\int_{1}^{t} 4du + 5 \right) \\ &= t \left(4t \Big|_{1}^{t} + 5 \right) \\ &= t(4t - 4) + 5t \\ &= 4t^2 + t. \end{aligned}$$

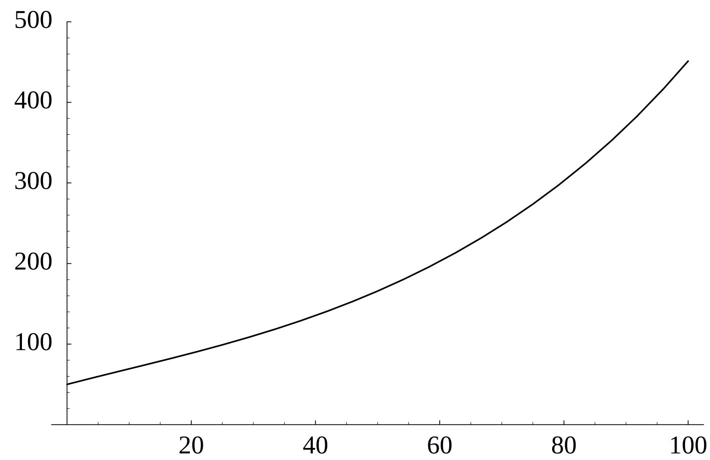

Figure 8.3.1 A solution of $\dot{x} = 0.02(1 + e^{-0.1t})x$

Example The equation

$$\dot{x} = p(t)x$$

with $x(0) = x_0$ may be used as a model for growth or decay where the rate of growth or decay is not necessarily a constant. Such an equation may be solved by separating variables, but the solution also follows from (8.3.11): Since $q(t) = 0$ for all t, we have

$$x = x_0 e^{\int_0^t p(s)ds}.$$

For example, if $p(t) = k$ for all t, where k is a constant, then we obtain the familiar solution

$$x = x_0 e^{kt}.$$

If $p(t) = 0.02(1 + e^{-0.1t})$, as in an example in Section 8.1, then

$$\begin{aligned}\int_0^t p(s)ds &= \int_0^t 00.02(1 + e^{-0.1s})ds \\ &= 0.02(s - 10e^{-0.1s})\Big|_0^t \\ &= 0.02(t - 10e^{-0.1t}) - 0.02(-10) \\ &= 0.02t - 0.02e^{-0.1t} + 0.2.\end{aligned}$$

Hence

$$x = x_0 e^{0.02t - 0.02e^{-0.1t} + 0.2.}.$$

The graph of x when $x_0 = 50$ is shown in Figure 8.3.1. You should compare this with the plot of an approximate solution in Figure 8.1.2.

Example A small reservoir holds 10,000 cubic feet of water. Water flows in at a rate of 100 cubic feet per hour and out at the same rate. Suppose initially the water in the reservoir contains 5 grams of salt per cubic foot, but the water flowing in contains 10 grams

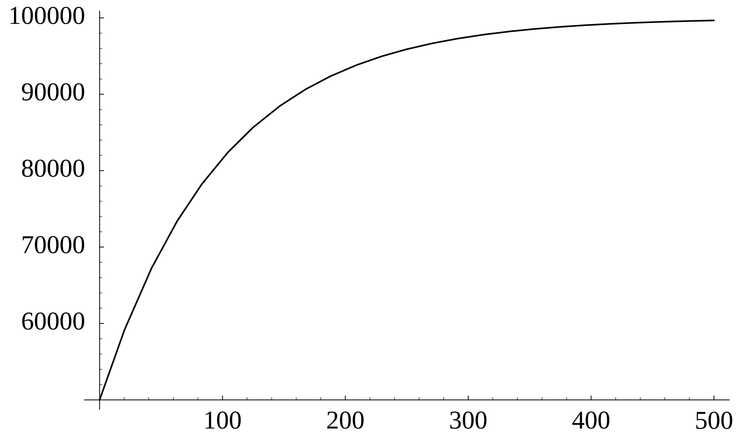

Figure 8.3.2 Graph of $x = 100,000 - 50,000e^{-0.01t}$

of salt per cubic foot. Let $x(t)$ be the amount of salt in the reservoir after t hours. Note that salt is entering the reservoir at a rate of 1000 grams per hour. Assuming the water in the reservoir is well-mixed at all times, the concentration of salt in the reservoir at time t is

$$\frac{x(t)}{10,000}$$

grams per cubic foot, from which it follows that salt is leaving the reservoir at a rate of

$$100\frac{x(t)}{10,000} = \frac{x(t)}{100}$$

grams per hour. Thus the rate of change of salt in the reservoir is given by

$$\dot{x} = 1000 - 0.01x.$$

That is, x satisfies a linear differential equation with $p(t) = -0.01$ and $q(t) = 1000$. Then

$$P(t) = -\int_0^t 0.01ds = -0.01t,$$

and so, using $x(0) = (5)(10,000) = 50,000$ grams, we have

$$\begin{aligned} x &= e^{-0.01t}\left(\int_0^t 1000e^{0.01u}du + 50,000\right) \\ &= e^{-0.01t}\left(100,000e^{0.01u}\Big|_0^t + 50,000\right) \\ &= e^{-0.01t}(100,000e^{0.01t} - 100,000 + 50,000) \\ &= 100,000 - 50,000e^{-0.01t}. \end{aligned}$$

In particular, note that

$$\lim_{t\to\infty} x(t) = \lim_{t\to\infty} 100,000 - 50,000e^{-0.01t} = 100,000,$$

and we see that over time the concentration of salt in the reservoir will approach the concentration of salt in its intake water. The graph of x is shown in Figure 8.3.2.

Problems

1. Solve each of the following linear differential equations.

   (a) $\dot{x} = 3x + 2t$, $x(0) = 2$
   (b) $\dot{x} = 2x - t^2$, $x(0) = 1$
   (c) $\dot{y} = 0.4y + 3$, $y(0) = 5$
   (d) $\dot{w} = -w + e^{-2t}$, $w(0) = 3$
   (e) $\dot{x} = \dfrac{2x}{t} + t^2$, $x(1) = 4$
   (f) $\dot{y} = -y + 2e^{-t} + t^2$, $y(0) = 1$

2. In 1990 the population of Botswana was 1.2 million. If $x(t)$ is the population of Botswana t years after 1990, suppose x satisfies the differential equation

$$\dot{x} = k(t)x,$$

   where $k(t)$ represents the rate of growth of the population at time t. At the start of 1990 the population of Botswana was growing at the rate of 2.9%, so $k(0) = 0.029$.

   (a) Suppose the rate of growth of the population is decreasing toward 1.5% in such a way that

$$k(t) = 0.015(1 + 0.93e^{0.04t}).$$

   Solve for x.

   (b) Compare your result in (a) with the result for a constant rate of growth of $k(t) = 0.029$ by plotting both solutions together.

3. Suppose the population of a certain country was 56 million in 2000 and the natural rate of the growth of the population was 2% per year. Moreover, suppose $k(t)$ represents the net rate of growth of the population due to immigration and emigration t years after 2000.

   (a) Let $y(t)$ be the population of the country t years after 2000. Explain why y should satisfy the differential equation

$$\frac{dy}{dt} = 0.02y + k(t),$$

   with $y(0) = 56$.

   (b) Solve the equation if $k(t) = 0.04t$. Plot your results.

   (c) What does this model predict for the population of the country in the year 2010?

   (d) When will the population of the country reach 100 million?

   (e) Compare your results with the numerical results obtained for the same problem in Problem 9 of Section 8.1.

4. A 500 gallon tank is initially filled with water with a concentration of 4 grams of salt per gallon. Water flows into the tank at the rate of 10 gallons per minute and is drawn off at the same rate. The concentration of salt in the intake water is 2 grams per gallon. Assume that the water in the tank is well-mixed at all times.

(a) Let $x(t)$ be the amount of salt in the tank at time t. Find a linear differential equation for x which models this situation.

(b) Solve the equation from (a) and graph the solution. What happens to x as t increases?

5. Suppose a tank holds V liters of a liquid which contains a certain chemical at a concentration of k_0 grams per liter. Liquid flows into the tank at rate of q liters per second and is drawn off at the same rate. The concentration of the chemical in the intake liquid is k grams per liter.

   (a) If $x(t)$ is the amount of the chemical in the tank at time t, show that x satisfies the linear differential equation

   $$\dot{x} = qk - \frac{q}{V}x$$

   with $x(0) = k_0 V$.

   (b) Solve the equation in (a). What happens to x as $t \to \infty$?

6. An equation of the form

   $$\dot{x} = p(t)x + q(t)x^n \tag{8.3.12}$$

   is called a *Bernoulli equation.* Note that the equation is linear if either $n = 0$ or $n = 1$.

   (a) Assume $n \neq 0$ and $n \neq 1$. Show that the substitution $w = x^{1-n}$ in (8.3.12) results in the linear differential equation

   $$\dot{w} = (1-n)p(t)w + (1-n)q(t).$$

   (b) Use the result of (a) to solve the equation

   $$\dot{x} = \frac{x}{t} - x^2$$

   with $x(1) = 1$.

   (c) Use the result of (a) to solve the equation

   $$\dot{=}x(1-x)$$

   with $x(0) = 0.5$.

7. Note that (c) of Problem 6 is a particular example of the logistic differential equation that we studied in Section 6.3 in our discussion of the inhibited population growth model. In general, we considered the logistic equation

   $$\dot{x} = \frac{\alpha}{M}x(M-x)$$

   with $x(0) = x_0$, where $x(t)$ is the size of the population at time t, α is the natural growth rate of the population, and M is the maximum size of the population that is sustainable in the given environment. Write this equation in the form of a Bernoulli equation and use the result from (a) of Problem 6 to show that

$$x = \frac{M}{1 + \beta e^{-\alpha t}}$$

where

$$\beta = \frac{M - x_0}{x_0}.$$