Section 4.3

The Fundamental Theorem of Calculus

We are now ready to make the long-promised connection between differentiation and integration, between areas and tangent lines. We will look at two closely related theorems, both of which are known as the Fundamental Theorem of Calculus. We will call the first of these the *Fundamental Theorem of Integral Calculus*.

Suppose f is integrable on $[a, b]$ and F is an antiderivative of f on (a, b) which is continuous on $[a, b]$. In particular, $F'(x) = f(x)$ for all x in (a, b). Let $P = \{x_0, x_1, x_2, \ldots, x_n\}$ be a partition of $[a, b]$ and, as usual, let $\Delta x_i = x_i - x_{i-1}$, $i = 1, 2, 3, \ldots, n$. Now

$$\begin{aligned} F(b) - F(a) &= F(x_n) - F(x_0) \\ &= F(x_n) + (F(x_{n-1}) - F(x_{n-1})) + (F(x_{n-2}) - F(x_{n-2})) + \cdots \\ &\qquad + (F(x_1) - F(x_1)) - F(x_0) \\ &= (F(x_n) - F(x_{n-1})) + (F(x_{n-1}) - F(x_{n-2})) + \cdots \\ &\qquad + (F(x_1) - F(x_0)) \\ &= \sum_{i=1}^{n} (F(x_i) - F(x_{i-1})). \end{aligned} \tag{4.3.1}$$

By the Mean Value Theorem, for every $i = 1, 2, 3, \ldots, n$, there exists a point c_i in the interval $[x_{i-1}, x_i]$ such that

$$F'(c_i) = \frac{F(x_i) - F(x_{i-1})}{x_i - x_{i-1}}. \tag{4.3.2}$$

Since $F'(c_i) = f(c_i)$ and $x_i - x_{i-1} = \Delta x_i$, it follows that

$$F(x_i) - F(x_{i-1}) = f(c_i)\Delta x_i. \tag{4.3.3}$$

Hence, putting (4.3.3) into (4.3.1),

$$F(b) - F(a) = \sum_{i=1}^{n} f(c_i)\Delta x_i. \tag{4.3.4}$$

Thus $F(b) - F(a)$ is equal to the value of a Riemann sum using the partition P, and so must lie between the upper and lower sums for P. That is, we have shown that for any partition P,

$$L(f, P) \le F(b) - F(a) \le U(f, P). \tag{4.3.5}$$

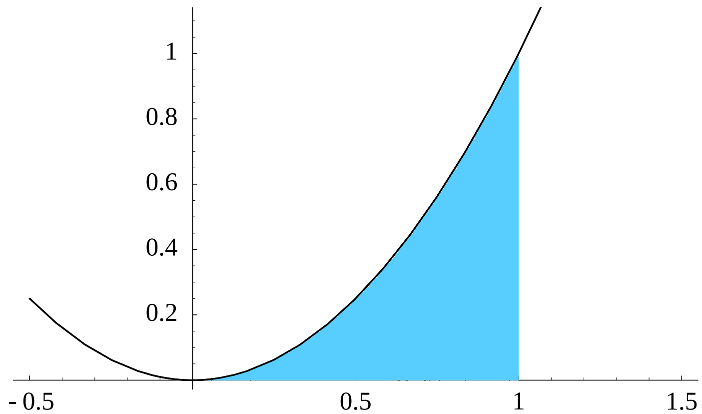

Figure 4.3.1 Region beneath the graph of $f(x) = x^2$ over the interval $[0, 1]$

But, since f is integrable, there is only one number that has this property, namely, $\int_a^b f(x)dx$. In other words, we have shown that

$$\int_a^b f(x)dx = F(b) - F(a). \tag{4.3.6}$$

Fundamental Theorem of Integral Calculus If f is integrable on $[a, b]$ and F is an antiderivative of f on (a, b) which is continuous on $[a, b]$, then

$$\int_a^b f(x)dx = F(b) - F(a). \tag{4.3.7}$$

This result reveals a sense in which integration is the inverse of differentiation: The definite integral of a function f may be evaluated easily, using (4.3.7), provided we can find a function F whose derivative is f.

It is common to write

$$F(x)\Big|_a^b$$

for $F(b) - F(a)$. With this notation, (4.3.7) becomes

$$\int_a^b f(x)dx = F(x)\Big|_a^b. \tag{4.3.8}$$

Example Since

$$F(x) = \frac{1}{3}x^3$$

is an antiderivative of $f(x) = x^2$, we have

$$\int_0^1 x^2 dx = \frac{1}{3}x^3\Big|_0^1 = \frac{1}{3} - 0 = \frac{1}{3}.$$

Thus the area under the parabola $y = x^2$ and above the interval $[0, 1]$ on the x-axis is exactly $\frac{1}{3}$. See Figure 4.3.1.

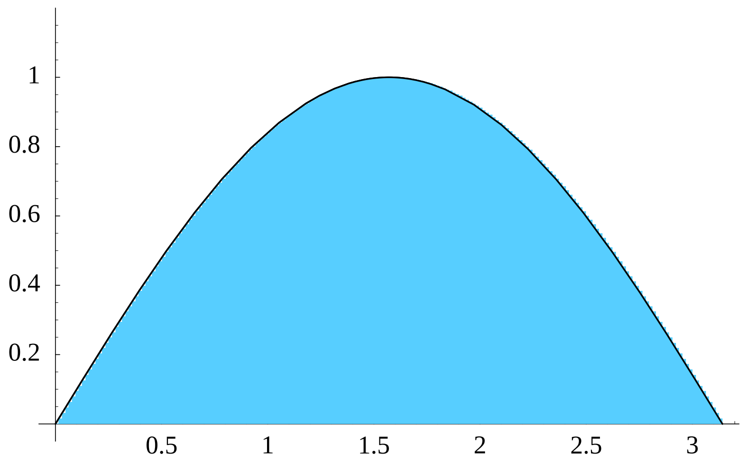

Figure 4.3.2 Region beneath the graph of $y = \sin(x)$ over the interval $[0, \pi]$

Note that F in the previous example is but one of an infinite number of antiderivatives of f. We can in fact use any antiderivative of f we want in applying (4.3.7), although we typically choose the simplest one we can find.

Example Since

$$G(x) = \frac{1}{3}x^3 + x$$

is an antiderivative of $g(x) = x^2 + 1$ (you may check by differentiating G), we have

$$\int_{-1}^{2} (x^2 + 1)dx = \left(\frac{1}{3}x^3 + x\right)\Bigg|_{-1}^{2} = \left(\frac{8}{3} + 2\right) - \left(-\frac{1}{3} - 1\right) = 6,$$

as we claimed in Section 4.2.

Example If A is the area under one arch of the curve $y = \sin(x)$, then

$$A = \int_0^{\pi} \sin(x)dx.$$

Since $F(x) = -\cos(x)$ is an antiderivative of $f(x) = \sin(x)$, we have

$$A = \int_0^{\pi} \sin(x)dx = -\cos(x)\Big|_0^{\pi} = -\cos(\pi) - (-\cos(0)) = 1 + 1 = 2.$$

See Figure 4.3.2.

Example Since

$$F(x) = \frac{4}{3}x^3 - \frac{1}{2}x^2 + 2x$$

is an antiderivative of $f(x) = 4x^2 - x + 2$ (again, you may check this by differentiating F), we have

$$\int_{-2}^{3} (4x^2 - x + 2)dx = \left(\frac{4}{3}x^3 - \frac{1}{2}x^2 + 2x\right)\Bigg|_{-2}^{3}$$
$$= \left(\frac{108}{3} - \frac{9}{2} + 6\right) - \left(-\frac{32}{3} - 2 - 4\right)$$
$$= \frac{325}{6}.$$

Example Since

$$F(t) = \frac{2}{3}t^{\frac{3}{2}}$$

is an antiderivative of $f(t) = \sqrt{t}$, we have

$$\int_{0}^{4} \sqrt{t}\, dt = \frac{2}{3}t^{\frac{3}{2}}\Bigg|_{0}^{4} = \frac{16}{3} - 0 = \frac{16}{3}.$$

As can be seen from these examples, the Fundamental Theorem of Integral Calculus provides us with a powerful tool for evaluating definite integrals exactly. However, to utilize the theorem we must first find an antiderivative for the function we are integrating. This turns out to be a difficult problem in general, and we will devote the next two sections, as well as parts of Chapter 6, to developing techniques to aid in finding antiderivatives. For example,

$$F(x) = -\frac{1}{2}x^3\cos(2x) + \frac{3}{4}x^2\sin(2x) + \frac{3}{4}x\cos(2x) - \frac{3}{8}\sin(2x)$$

is an antiderivative of $f(x) = x^3\sin(2x)$, as may be checked by differentiation, but at this point it is not clear how to find such an antiderivative in the first place. Moreover, there are integrable functions, even relatively simple ones such as

$$f(x) = \frac{\sin(x)}{x},$$

which do not have antiderivatives expressible in terms of the elementary functions studied in calculus.

The Fundamental Theorem of Integral Calculus tells us that if a function f has an antiderivative, then we may use that antiderivative to evaluate a definite integral of f, but it does not tell us which functions have antiderivatives. The Fundamental Theorem of Differential Calculus will tell us, in part, that every continuous function has an antiderivative. Before beginning that discussion, we need to extend the definition of the definite integral slightly.

The definition of $\int_a^b f(x)dx$ in Section 4.1 implicitly assumes that $a < b$. For the work we are about to do, we need to extend the definition to include $a \geq b$, as we did in Problem 9 of Section 4.1. First of all, if $a = b$, it would seem reasonable for the value of the definite integral to be 0 since the region between the graph of the function and the x-axis has been reduced to a line segment. Hence we make the following definition.

Definition For any function f defined at a point a, we define

$$\int_a^a f(x)dx = 0. \tag{4.3.9}$$

Note that with this definition, the statement

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx, \tag{4.3.10}$$

which we discussed in Section 4.1 in the case $a < c < b$, holds true even if $a = c$, $b = c$, or $a = b = c$. Now suppose we have $a < b < c$. Then

$$\int_a^c f(x)dx = \int_a^b f(x)dx + \int_b^c f(x)dx, \tag{4.3.11}$$

from which it follows that

$$\int_a^b f(x)dx = \int_a^c f(x)dx - \int_b^c f(x)dx. \tag{4.3.12}$$

If we define

$$\int_c^b f(x)dx = -\int_b^c f(x)dx, \tag{4.3.13}$$

then we may rewrite (4.3.12) in the form of (4.3.10). For this reason, we make the following definition.

Definition If $b < a$ and f is integrable on $[b, a]$, we define

$$\int_a^b f(x)dx = -\int_b^a f(x)dx. \tag{4.3.14}$$

You may check that with these two extensions to the definition of the definite integral, we may now state the following proposition.

Proposition If f is integrable on a closed interval containing the points a, b, and c, then

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx. \tag{4.3.15}$$

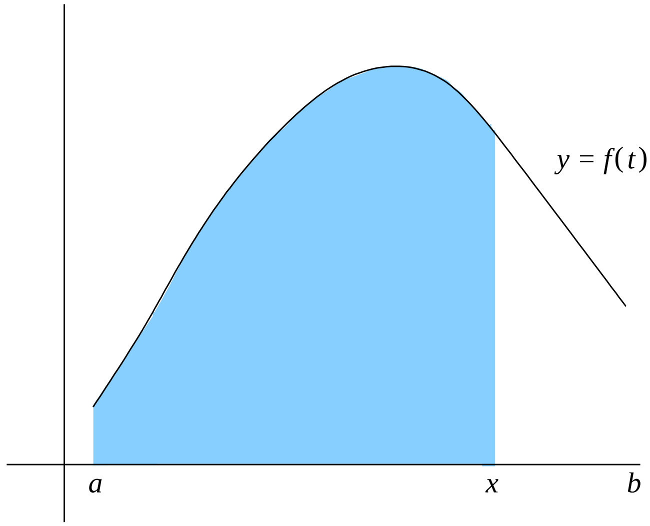

Figure 4.3.3 $F(x) = \int_a^x f(t)dt$ is the area from a to x

We may now return to our discussion of antiderivatives and the Fundamental Theorem of Differential Calculus. Suppose f is continuous on the interval $[a, b]$. We want to construct an antiderivative for f on (a, b). From the Fundamental Theorem of Integral Calculus, we know that if F is an antiderivative of f on (a, b) which is continuous on $[a, b]$, then for any x in (a, b) we would have

$$\int_a^x f(t)dt = F(x) - F(a), \tag{4.3.16}$$

that is,

$$F(x) = F(a) + \int_a^x f(t)dt. \tag{4.3.17}$$

Hence, if we are seeking an antiderivative for f, it makes sense to define

$$F(x) = \int_a^x f(t)dt \tag{4.3.18}$$

and verify that $F'(x) = f(x)$ for all x in (a, b). Note that $F(x)$, geometrically, is the cumulative area between the graph of f and the x-axis from a to x, as shown in Figure 4.3.3. We need to compute

$$F'(x) = \lim_{h\to 0} \frac{F(x+h) - F(x)}{h} = \lim_{h\to 0} \frac{1}{h}\left(\int_a^{x+h} f(t)dt - \int_a^x f(t)dt\right) \tag{4.3.19}$$

for x in (a, b). Now

$$\int_a^{x+h} f(t)dt = \int_a^x f(t)dt + \int_x^{x+h} f(t)dt, \tag{4.3.20}$$

so

$$\int_a^{x+h} f(t)dt - \int_a^x f(t)dt = \int_x^{x+h} f(t)dt. \tag{4.3.21}$$

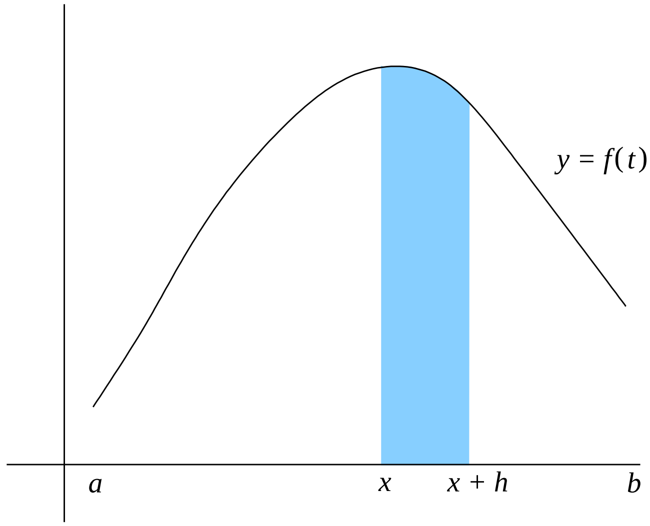

Figure 4.3.4 $\int_a^{x+h} f(t)dt - \int_x^a f(t)dt = \int_x^{x+h} f(t)dt$

See Figure 4.3.4. Thus

$$F'(x) = \lim_{h \to 0} \frac{1}{h} \int_x^{x+h} f(t)dt. \tag{4.3.22}$$

Suppose $h > 0$. Since f is continuous, f has a minimum value $m(h)$ and a maximum value $M(h)$ on the interval $[x, x+h]$. Hence $m(h) \leq f(x) \leq M(h)$ for all x in $[x, x+h]$, from which it follows that

$$\int_x^{x+h} m(h)dt \leq \int_x^{x+h} f(t)dt \leq \int_x^{x+h} M(h)dt. \tag{4.3.23}$$

Since $m(h)$ and $M(h)$ are constants, (4.3.24) implies

$$m(h)h \leq \int_x^{x+h} f(t)dt \leq M(h)h. \tag{4.3.24}$$

Thus

$$m(h) \leq \frac{1}{h} \int_x^{x+h} f(t)dt \leq M(h). \tag{4.3.25}$$

Now $m(h) = f(c)$ for some c in $[x, x+h]$. Moreover, as h approaches 0, $x+h$ approaches x, and so c must also approach x. Hence, since f is continuous,

$$\lim_{h \to 0^+} m(h) = \lim_{h \to 0^+} f(c) = f(x). \tag{4.3.26}$$

Similarly,

$$\lim_{h \to 0^+} M(h) = f(x). \tag{4.3.27}$$

It now follows from (4.3.25) that

$$\lim_{h \to 0^+} \frac{1}{h} \int_x^{x+h} f(t)dt = f(x). \tag{4.3.28}$$

A similar argument shows that

$$\lim_{h \to 0^-} \frac{1}{h} \int_x^{x+h} f(t)dt = f(x), \tag{4.3.29}$$

and so we may conclude that

$$F'(x) = \lim_{h \to 0} \frac{1}{h} \int_x^{x+h} f(t)dt = f(x). \tag{4.3.30}$$

That is,

$$F(x) = \int_a^x f(t)dt$$

is an antiderivative of f on (a, b).

Fundamental Theorem of Differential Calculus If f is continuous on the closed interval $[a, b]$ and F is the function on (a, b) defined by

$$F(x) = \int_a^x f(t)dt, \tag{4.3.31}$$

then F is differentiable on (a, b) with $F'(x) = f(x)$ for all x in (a, b). In other words,

$$\frac{d}{dx} \int_a^x f(t)dt = f(x) \tag{4.3.32}$$

for all x in (a, b).

It is worth noting that (4.3.32) holds for $x < a$ as well, as long as f is continuous on a closed interval which contains both x and a.

Example Let

$$f(x) = \begin{cases} \dfrac{\sin(x)}{x}, & \text{if } x \neq 0, \\ 1, & \text{if } x = 0. \end{cases}$$

Then f is continuous on $(-\infty, \infty)$, so

$$F(x) = \int_0^x f(t)dt$$

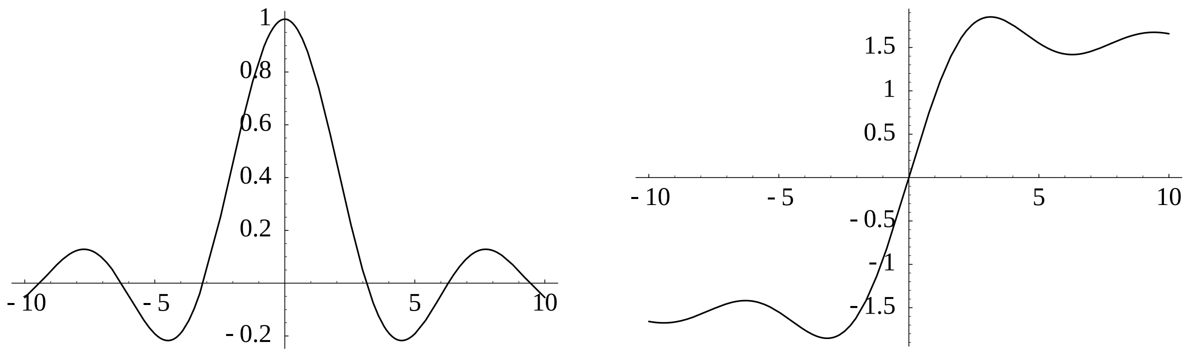

Figure 4.3.5 Graphs of $f(x) = \dfrac{\sin(x)}{x}$ and $F(x) = \displaystyle\int_0^x \frac{\sin(t)}{t}\,dt$

is an antiderivative of f on $(-\infty, \infty)$. In particular,

$$F'(x) = \frac{\sin(x)}{x}$$

for all $x \neq 0$. The graphs of F and f are shown in Figure 4.3.5. Geometrically, $F(x)$ is the cumulative area between the graph of f and the x-axis from 0 to x and $F'(x)$ is the rate at which area is accumulating as x increases. Since the rate at which area is accumulating depends on the height of the curve, it is natural to expect, and the Fundamental Theorem of Differential Calculus confirms, that $F'(x) = f(x)$. The function F is known as the *sine integral function*. It may be shown that it is not representable in closed form in terms of the elementary functions of calculus.

Example Using Leibniz notation,

$$\frac{d}{dx}\int_0^x \sin(t^2)dt = \sin(x^2).$$

Example Suppose

$$G(x) = \int_0^{3x} \sin(t^2)dt.$$

Then $G(x) = F(h(x))$, where $h(x) = 3x$ and

$$F(x) = \int_0^x \sin(t^2)dt.$$

Hence, using the chain rule,

$$G'(x) = F'(h(x))h'(x) = \sin((3x)^2)(3) = 3\sin(9x^2).$$

Example Suppose

$$H(x) = \int_x^0 \frac{1}{1+t^4}\,dt.$$

Then, using (4.3.14),

$$H(x) = -\int_0^x \frac{1}{1+t^4}\, dt,$$

so

$$H'(x) = -\frac{d}{dx}\int_0^x \frac{1}{1+t^4}\, dt = -\frac{1}{1+x^2}.$$

Example Suppose

$$F(x) = \int_{2x}^{x^2} \sqrt{1+t^4}\, dt.$$

Then, using (4.3.15) and (4.3.14),

$$F(x) = \int_{2x}^{0} \sqrt{1+t^4}\, dt + \int_0^{x^2} \sqrt{1+t^4}\, dt = -\int_0^{2x} \sqrt{1+t^4}\, dt + \int_0^{x^2} \sqrt{1+t^4}\, dt.$$

Note that there is nothing special about using 0 in this decomposition, other than the requirement that the function $f(t) = \sqrt{1+t^4}$ be integrable on all of the relevant intervals. Now we have

$$\begin{aligned} F'(x) &= -\frac{d}{dx}\int_0^{2x} \sqrt{1+t^4}\, dt + \frac{d}{dx}\int_0^{x^2} \sqrt{1+t^4}\, dt \\ &= -\sqrt{1+(2x)^4}\,(2) + \sqrt{1+(x^2)^4}\,(2x) \\ &= 2x\sqrt{1+x^8} - 2\sqrt{1+16x^4}. \end{aligned}$$

To summarize this section, the Fundamental Theorem of Integral Calculus provides us with an elegant method for evaluating definite integrals, but is useful only when we can find an antiderivative for the function being integrated. The Fundamental Theorem of Differential Calculus tells us that every continuous function has an antiderivative and shows how to construct one using the definite integral. Unfortunately, this brings us in circle and does not provide us with an effective means for finding antiderivatives to use in applying the Fundamental Theorem of Integral Calculus. For example, we know that

$$F(x) = \int_1^x \frac{\sin(t)}{t}\, dt$$

is an antiderivative of

$$f(x) = \frac{\sin(x)}{x},$$

but this is of no help in evaluating, say,

$$\int_1^4 \frac{\sin(x)}{x}\, dx.$$

Hence in order to fully utilize the Fundamental Theorem of Integral Calculus in the evaluation of definite integrals, we must develop some procedures to aid in finding antiderivatives. We will turn to this problem in the next section.

Problems

1. Evaluate the following definite integrals using the Fundamental Theorem of Integral Calculus.

   (a) $\int_0^1 x dx$
   (b) $\int_0^3 (x^2 + 2x) dx$
   (c) $\int_0^2 x^3 dx$
   (d) $\int_{-1}^1 x^3 dx$
   (e) $\int_0^2 (2x^3 + 3x^2 + x - 4) dx$
   (f) $\int_1^4 \frac{1}{x^2} dx$
   (g) $\int_1^4 \frac{1}{\sqrt{t}} dt$
   (h) $\int_0^8 \sqrt{t+1}\, dt$
   (i) $\int_0^{\frac{\pi}{2}} \sin(x) dx$
   (j) $\int_{-\pi}^{\pi} \cos(z) dz$

2. Evaluate the following definite integrals using the Fundamental Theorem of Integral Calculus.

   (a) $\int_0^2 (x+1)^2 dx$
   (b) $\int_0^2 (2x+1)^2 dx$
   (c) $\int_0^4 \sqrt{1+2t}\, dt$
   (d) $\int_0^{\frac{\pi}{4}} \sec^2(x) dx$
   (e) $\int_0^{\pi} \sin(2x) dx$
   (f) $\int_0^{\pi} 5\cos(3x) dx$
   (g) $\int_0^{\frac{\pi}{3}} 4\sin(3x) dx$
   (h) $\int_{-\pi}^{\pi} 8\cos(5\theta) d\theta$
   (i) $\int_0^{\sqrt{\pi}} 2x\sin(x^2) dx$
   (j) $\int_{-1}^2 2x(1+x^2)^5 dx$
   (k) $\int_0^{\sqrt{\pi}} x\sin(x^2) dx$
   (l) $\int_{-1}^2 2x(1+x^2)^5 dx$

3. For each of the following functions, graph both f and

$$F(x) = \int_0^x f(t) dt$$

   together over the given interval.

   (a) $f(x) = \sin(x)$ on $[-2\pi, 2\pi]$
   (b) $f(x) = \sin(x^2)$ on $[0, 10]$
   (c) $f(x) = \frac{1}{1+x^4}$ on $[-3, 3]$
   (d) $f(x) = \frac{1}{x+1}$ on $[0, 10]$

4. Find the derivatives of each of the following functions.

   (a) $F(x) = \int_0^x \sin^2(4t)dt$

   (b) $g(x) = \int_2^x \frac{3}{t+2}\, dt$

   (c) $F(x) = \int_x^\pi \cos^3(t)dt$

   (d) $G(t) = \int_t^0 \sqrt{4 - z^2}\, dz$

   (e) $f(x) = \int_0^{x^2} \frac{1}{1+s^2}\, ds$

   (f) $h(z) = \int_z^{3z} \sqrt{1+t^2}\, dt$

5. Evaluate the following derivatives.

   (a) $\frac{d}{dx} \int_1^x \frac{1}{1+t^2}\, dt$

   (b) $\frac{d}{dx} \int_1^{3x} \frac{\sin(3t)}{t}\, dt$

   (c) $\frac{d}{dt} \int_{t^2}^5 \sin^2(3x)dx$

   (d) $\frac{d}{dx} \int_{3x}^{x^2} \frac{1}{\sqrt{1+t^2}}\, dt$

6. Find the area of the region beneath one arch of the curve $y = 3\sin(2x)$.

7. Let R be the region bounded by the curves $y = x^2$ and $y = (x-2)^2$ and the x-axis. Find the area of R.

8. Explain why the integral

$$\int_0^1 (x - x^2)dx$$

   is the area of the region bounded by the curves $y = x^2$ and $y = x$. Find this area.

9. Explain why the integral

$$\int_{-1}^1 (2 - 2x^2)dx$$

   is the area of the region bounded by the curves $y = 1 - x^2$ and $y = x^2 - 1$. Find this area.