D
ifference Equations
to
ifferential Equations

In this section we will look at several applications of the exponential and logarithm functions to problems involving growth and decay, including compound interest, radioactive decay, and population growth.

Compound interest

Suppose a principal of P dollars is deposited in a bank which pays $100i\%$ interest compounded n times a year. That is, each year is divided into n units and after each unit of time the bank pays $\frac{100i}{n}\%$ interest on all money currently in the account, including money that was earned as interest at an earlier time. Thus if x_m represents the amount of money in the account after m units of time, x_m must satisfy the difference equation

$$x_{m+1} - x_m = \frac{i}{n} x_m, \tag{6.3.1}$$

$m = 0, 1, 2, \ldots$, with initial condition $x_0 = P$. Hence the sequence $\{x_m\}$ satisfies the linear difference equation

$$x_{m+1} = \left(1 + \frac{i}{n}\right) x_m, \tag{6.3.2}$$

and so, from our work in Section 1.4, we know that

$$x_m = \left(1 + \frac{i}{n}\right)^m x_0 = \left(1 + \frac{i}{n}\right)^m P \tag{6.3.3}$$

for $m = 0, 1, 2, \ldots$. If we let $A(t)$ be the amount in the account after t years, then, since there are nt compounding periods in t years,

$$A(t) = x_{nt} = \left(1 + \frac{i}{n}\right)^{nt} P \tag{6.3.4}$$

Example   Suppose $1,000 is deposited at 5% interest which is compounded quarterly. If $A(t)$ is the amount in the account after t years, then, for example,

$$A(5) = 1000 \left(1 + \frac{0.05}{4}\right)^{20} = 1,282.04,$$

rounded to the nearest cent. If the interest were compounded monthly instead, then we would have

$$A(5) = 1000 \left(1 + \frac{0.05}{12} \right)^{60} = 1,283.36.$$

Of course, the more frequent the compounding, the faster the amount in the account will grow. At the same time, there is no limit to how often the bank could compound. However, is there some limit to how fast the account can grow? That is, for a fixed value of t, is $A(t)$ bounded as n grows? To answer this question, we need to consider

$$\lim_{n \to \infty} \left(1 + \frac{i}{n} \right)^{nt}.$$

To evaluate this limit, first consider the limit

$$\lim_{x \to \infty} \left(1 + \frac{k}{x} \right)^{x},$$

where k is a constant. If we let

$$y = \left(1 + \frac{k}{x} \right)^{x},$$

then

$$\log(y) = x \log \left(1 + \frac{k}{x} \right).$$

Using l'Hôpital's rule, we have

$$\begin{aligned}
\lim_{x \to \infty} \log(y) &= \lim_{x \to \infty} x \log \left(1 + \frac{k}{x} \right) \\
&= \lim_{x \to \infty} \frac{\log \left(1 + \frac{k}{x} \right)}{\frac{1}{x}} \\
&= \lim_{x \to \infty} \frac{\frac{d}{dx} \log \left(1 + \frac{k}{x} \right)}{\frac{d}{dx} \left(\frac{1}{x} \right)} \\
&= \lim_{x \to \infty} \frac{\frac{1}{1 + \frac{k}{x}} \left(-\frac{k}{x^2} \right)}{-\frac{1}{x^2}} \\
&= \lim_{x \to \infty} \frac{k}{1 + \frac{k}{x}} \\
&= k.
\end{aligned}$$

It now follows that

$$\lim_{x \to \infty} y = \lim_{x \to \infty} e^{\log(y)} = e^{k}.$$
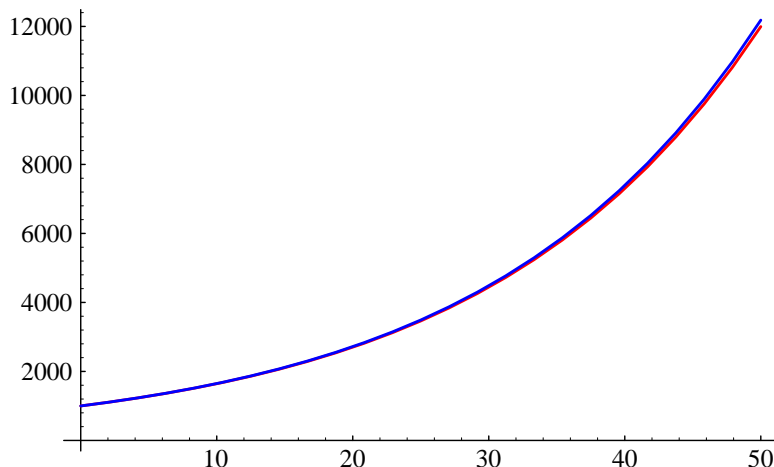
That is, we have the following proposition.

Figure 6.3.1 Compounding quarterly versus compounding continuously (5% interest)

Proposition   For any constant k,

$$\lim_{x \to \infty} \left(1 + \frac{k}{x} \right)^x = e^k. \tag{6.3.5}$$

It now follows that

$$\lim_{n \to \infty} \left(1 + \frac{i}{n} \right)^{nt} = \left(\lim_{n \to \infty} \left(1 + \frac{i}{n} \right)^n \right)^t = (e^i)^t = e^{it}. \tag{6.3.6}$$

Hence

$$\lim_{n \to \infty} A(t) = \lim_{n \to \infty} P \left(1 + \frac{i}{n} \right)^{nt} = Pe^{it}.$$

Thus no matter how many times interest is compounded per year, the amount after t years will never exceed Pe^{it}. We think of Pe^{it} as the amount that would be in the account if interest were *compounded continuously*.

Example   In the previous example, with $P = \$1,000$ and $i = 0.05$, the amount after five years of interest compounded continuously would be

$$1000e^{(0.05)(5)} = 1,284.03.$$

In other words, assuming a 5% interest rate, no matter how many times per year the bank compounds the interest, the amount in the account after five years can never exceed $1,284.03. As Figure 6.3.1 shows, in this case there is only a slight difference between compounding quarterly and compounding continuously over a period of 50 years.

Growth and decay

We saw in Chapter 1 that the linear difference equation

$$x_{n+1} - x_n = \alpha x_n \tag{6.3.7}$$

may be used as as simple model for the growth of a population when $\alpha > 0$ or as a model for radioactive decay when $\alpha < 0$. As we discussed in Section 6.1, the continuous time version of this model is the differential equation

$$\dot{x}(t) = \alpha x(t). \tag{6.3.8}$$

At that time we saw that the solution of this equation is given by

$$x(t) = x_0 e^{\alpha t}, \tag{6.3.9}$$

where $x_0 = x(0)$. As before, when $\alpha > 0$ this is a model for uninhibited, also called natural, population growth, while when $\alpha < 0$ it is a model for radioactive decay. More generally, this model is applicable whenever a quantity is known to change at a rate proportional to itself, as expressed by (6.3.8).

Example   Suppose the population of a certain country was 23 million in 1990 and 27 million in 1995. Assuming an uninhibited population growth model, if $x(t)$ represents the size of the population, in millions, t years after 1990, then

$$x(t) = 23e^{\alpha t}$$

for some value of α. To find α, we note that

$$27 = x(5) = 23e^{5\alpha}.$$

Hence

$$e^{5\alpha} = \frac{27}{23},$$

from which we obtain

$$5\alpha = \log\left(\frac{27}{23}\right).$$

Thus

$$\alpha = \frac{1}{5}\log\left(\frac{27}{23}\right) = 0.0321,$$

where we have rounded to four decimal places. Hence

$$x(t) = 23e^{0.0321t}.$$

For example, this model would predict a population in 2000 of

$$x(10) = 23e^{(0.0321)(10)} = 31.7 \text{ million.}$$

Also, assuming this model continues to be valid, we could compute how many years it would take for the population to reach any given size. For example, if T is the number of years until the population doubles, then we would have

$$46 = x(T) = 23e^{0.0321T}.$$

Thus

$$e^{0.0321T} = 2,$$

so

$$0.0321T = \log(2)$$

and

$$T = \frac{\log(2)}{0.0321} = 21.6,$$

to one decimal place. Hence a population growing at this rate will double in size in less than 22 years.

Example A common method for dating fossilized remains of animal and plant life is to compare the amount of carbon-14 to the amount of carbon-12 in the fossil. For example, the bones of a living animal contain approximately equal amounts of these two elements, but after death the carbon-14 begins to decay, whereas the carbon-12, not being radioactive, remains at a constant level. Hence it is possible to determine the age of the fossil from the amount of carbon-14 that remains. In particular, if $x(t)$ is the amount of carbon-14 in the fossil t years after the animal died, then

$$\dot{x}(t) = \alpha x(t)$$

for some constant α, and so

$$x(t) = x_0 e^{\alpha t},$$

where x_0 is the initial amount of carbon-14. Since it is known that the half-life of carbon-14 is 5,730 years (that is, one-half of any initial amount of carbon-14 will decay over a period of 5,730 years), we can find the value of α. Namely, we know that

$$\frac{1}{2} x_0 = x(5730) = x_0 e^{5730\alpha},$$

so

$$e^{5730\alpha} = \frac{1}{2}.$$

Hence

$$5730\alpha = -\log(2),$$

so

$$\alpha = -\frac{\log(2)}{5730}.$$

For example, suppose a fossilized bone is found which has 10% of its original carbon-14. If T is the time since the death of the animal, we must have

$$\frac{1}{10} x_0 = x(T) = x_0 e^{\alpha T}.$$

Thus

$$e^{\alpha T} = \frac{1}{10},$$

so

$$\alpha T = -\log(10)$$

and

$$T = -\frac{\log(10)}{\alpha} = 5730\frac{\log(10)}{\log(2)} = 19,035 \text{ years},$$

rounding to the nearest year. Hence the fossil is from an animal that died more than 19,000 years ago.

Inhibited growth models

In Section 1.5 we discussed a modification of the uninhibited growth model which took into account the limits placed on growth by environmental factors. In this model, which we called the inhibited growth model, if x_n is the size of the population after n units of time, α is the natural growth rate of the population (that is, the rate of growth the population would experience if it were not for the limiting factors), and M is the maximum population which is sustainable in the given environment, then

$$x_{n+1} - x_n = \alpha x_n \left(\frac{M - x_n}{M}\right) \tag{6.3.10}$$

for $n = 0, 1, 2, \ldots$. Hence this model modifies the natural rate of growth by the factor

$$\frac{M - x_n}{M}, \tag{6.3.11}$$

representing the proportion of room which is left for future growth. As a result, when x_n is small, (6.3.11) is close to 1 and the population grows at a rate close to its natural rate; however, as x_n increases toward M, (6.3.11) decreases, causing the rate of the growth of the population to decrease toward 0.

Now (6.3.10) says that the amount of increase in the population during one unit of time is jointly proportional to the size of the population and the proportion of room left for growth. Thus for a continuous time model, if $x(t)$ is the size of the population at time t, then the rate of change of $x(t)$ should be jointly proportional to $x(t)$ and $\dfrac{M - x(t)}{M}$. That is, $x(t)$ should satisfy the differential equation

$$\dot{x}(t) = \alpha x(t)\left(\frac{M - x(t)}{M}\right) = \frac{\alpha}{M}x(t)(M - x(t)). \tag{6.3.12}$$

This equation is called the *logistic differential equation*. It has many applications other than population growth; for example, it is frequently used as a model for the spread of an infectious disease, where $x(t)$ represents the number of people who have contracted the disease by time t, M is the total size of the population that could potentially be infected, and α is a parameter controlling the rate at which the disease spreads.

To solve the logistic differential equation, we begin by rewriting (6.3.12) as

$$\frac{\dot{x}(t)}{x(t)(M - x(t))} = \frac{\alpha}{M}. \tag{6.3.13}$$

Since this is an equation involving the derivative of the function we are trying to find, we might try integrating as a step toward finding $x(t)$. That is, if we replace t by s in (6.3.13) and then integrate from 0 to t, we obtain

$$\int_0^t \frac{x(s)}{x(s)(M-x(s))}\, ds = \int_0^t \frac{\alpha}{M} = \frac{\alpha}{M}t. \qquad (6.3.14)$$

To evaluate the remaining integral in (6.3.14), we first make the substitution

$$u = x(s)$$
$$du = \dot{x}(s)ds.$$

Then, letting $x_0 = x(0)$, which we assume to be less than M, we have

$$\int_0^t \frac{\dot{x}(s)}{x(s)(M-x(s))}\, ds = \int_{x_0}^{x(t)} \frac{1}{u(M-u)}\, du. \qquad (6.3.15)$$

To evaluate this integral, we use the algebraic fact, known as *partial fraction decomposition*, that there exist constants A and B such that

$$\frac{1}{u(M-u)} = \frac{A}{u} + \frac{B}{M-u}. \qquad (6.3.16)$$

Once we find the values for A and B, the integration will follow easily. Now (6.3.16) implies that

$$\frac{1}{u(M-u)} = \frac{A(M-u)+Bu}{u(M-u)}.$$

Since two rational functions with equal denominators are equal only if their numerators are also equal, it follows that

$$1 = A(M-u) + Bu.$$

This final equality must hold for all values of u, so, in particular, when $u = 0$ we obtain

$$1 = AM$$

and when $u = M$ we have

$$1 = BM.$$

It follows that

$$A = \frac{1}{M}$$

and

$$B = \frac{1}{M}.$$

Hence

$$\frac{1}{u(M-u)} = \frac{1}{M}\frac{1}{u} + \frac{1}{M}\frac{1}{M-u},$$

and so

$$\int_{x_0}^{x(t)} \frac{1}{u(M-u)} \, du = \frac{1}{M} \int_{x_0}^{x(t)} \frac{1}{u} \, du + \frac{1}{M} \int_{x_0}^{x(t)} \frac{1}{M-u} \, du$$

$$= \left(\frac{1}{M} \log(u) - \frac{1}{M} \log(M-u) \right) \Bigg|_{x_0}^{x(t)}$$

$$= \frac{1}{M} \log \left(\frac{u}{M-u} \right) \Bigg|_{x_0}^{x(t)}$$

$$= \frac{1}{M} \log \left(\frac{x(t)}{M-x(t)} \right) - \frac{1}{M} \log \left(\frac{x_0}{M-x_0} \right)$$

$$= \frac{1}{M} \log \left(\left(\frac{x(t)}{M-x(t)} \right) \left(\frac{M-x_0}{x_0} \right) \right).$$

Here we have used the fact that $x(t) > 0$ for all t and the assumption that we are working with values of t for which $x(t) < M$ to avoid the need for absolute values. We will see below that in fact the latter assumption holds for all t. Combining with (6.3.14), we have

$$\frac{1}{M} \log \left(\left(\frac{x(t)}{M-x(t)} \right) \left(\frac{M-x_0}{x_0} \right) \right) = \frac{1}{M} t.$$

Multiply both sides by M and the applying the exponential function gives us

$$\left(\frac{x(t)}{M-x(t)} \right) \left(\frac{M-x_0}{x_0} \right) = e^{\alpha t}.$$

Letting

$$\beta = \frac{M-x_0}{x_0}, \tag{6.3.17}$$

we have

$$\beta x(t) = e^{\alpha t}(M - x(t)).$$

Hence

$$\beta x(t) + x(t)e^{\alpha t} = Me^{\alpha t},$$

so

$$(\beta + e^{\alpha t})x(t) = Me^{\alpha t}.$$

This gives us

$$x(t) = \frac{Me^{\alpha t}}{\beta + e^{\alpha t}},$$

or, after dividing through by $e^{\alpha t}$,

$$x(t) = \frac{M}{1 + \beta e^{-\alpha t}}. \tag{6.3.18}$$

Note that since $1 + \beta e^{-\alpha t} > 1$ for all t, we have, as we assumed above, $x(t) < M$ for all t. If we substitute back in the value for β, we have, finally,

$$x(t) = \frac{x_0 M}{x_0 + (M - x_0)e^{-\alpha t}}. \qquad (6.3.19)$$

Note that

$$\lim_{t \to \infty} x(t) = \lim_{t \to \infty} \frac{x_0 M}{x_0 + (M - x_0)e^{-\alpha t}} = \frac{x_0 M}{x_0} = M, \qquad (6.3.20)$$

showing that the population, although never exceeding M, will nevertheless approach M asymptotically.

Example The population of the United States was 179.3 million in 1960, 203.3 million in 1970, and 226.5 million in 1980. Let $x(t)$ represent the population, in millions, of the United States t years after 1960. To fit the logistic model to this data, we need to find constants α and M so that

$$x(t) = \frac{179.3M}{179.3 + (M - 179.3)e^{-\alpha t}}$$

for $t = 10$ and $t = 20$ (note that we already have $x(0) = 179.3$). That is, we need to solve the equations

$$203.3 = x(10) = \frac{179.3M}{179.3 + (M - 179.3)e^{-10\alpha}}$$

$$226.5 = x(20) = \frac{179.3M}{179.3 + (M - 179.3)e^{-20\alpha}}$$

for α and M. Working with the first equation, we have

$$(203.3)(179.3) + (203.3)(M - 179.3)e^{-10\alpha} = 179.3M,$$

which gives us

$$(203.3)(M - 179.3)e^{-10\alpha} = 179.3(M - 203.3).$$

Thus

$$e^{-10\alpha} = \frac{179.3(M - 203.3)}{203.3(M - 179.3)}. \qquad (6.3.21)$$

Similarly, the second equation gives us

$$e^{-20\alpha} = \frac{179.3(M - 226.5)}{226.5(M - 179.3)}.$$

Now

$$e^{-20\alpha} = \left(e^{-10\alpha}\right)^2,$$

so we have

$$\frac{179.3(M - 226.5)}{226.5(M - 179.3)} = \left(\frac{179.3(M - 203.3)}{203.3(M - 179.3)}\right)^2.$$
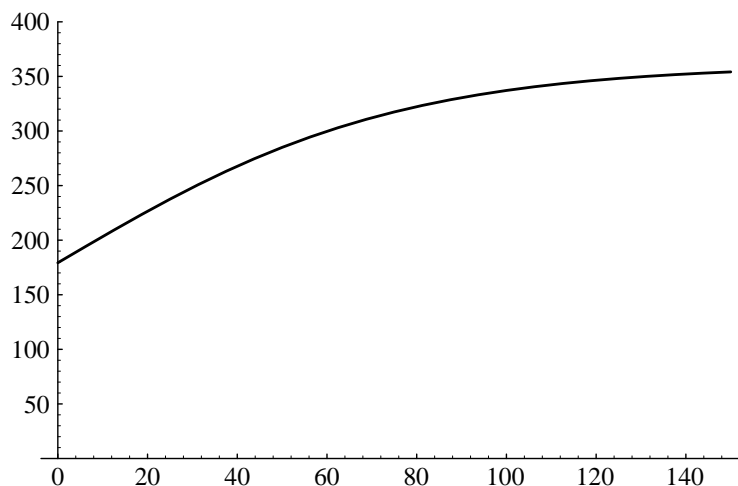
Figure 6.3.2 Inhibited growth model for the United States (1960-2110)

Thus

$$(203.3)^2(M - 226.5)(M - 179.3) = (179.3)(226.5)(M - 203.3)^2,$$

which, when expanded, gives us

$$(203.3)^2(M^2 - 405.8M + (179.3)(226.5)) = (179.3)(226.5)(M^2 - 406.6M + (203.3)^2).$$

Hence

$$719.44M^2 - 259,459.59M = 0.$$

Since $M \neq 0$, the desired solution must be

$$M = \frac{259,459.59}{719.44} = 360.6,$$

rounded to the first decimal place. Substituting this value for M into (6.3.21), we have

$$e^{-10\alpha} = \frac{179.3(360.6 - 203.3)}{203.3(360.6 - 179.3)},$$

and so

$$\alpha = -\frac{1}{10}\log\left(\frac{179.3(360.6 - 203.3)}{203.3(360.6 - 179.3)}\right) = 0.02676,$$

rounded to five decimal places. Thus we have

$$x(t) = \frac{(179.3)(360.6)}{179.3 + (360.6 - 179.3)e^{-0.02676t}} = \frac{64,655.6}{179.3 + 181.3e^{-0.02676t}}.$$

For example, this model would predict a population in 1990 of

$$x(30) = \frac{64,655.6}{179.3 + 181.3e^{-(0.02676)(30)}} = 248.2 \text{ million}$$

and a population in 2000 of

$$x(40) = \frac{64,655.6}{179.3 + 181.3e^{-(0.02676)(40)}} = 267.8 \text{ million}.$$

The 1990 prediction is very close to the actual population in 1990, which was approximately 249.6 million, and the prediction for the year 2000 is very close to the Census Bureau's prediction of 268.3 million. Recall that the uninhibited growth model for the United States, based on population data for 1970 and 1980, predicted a population of 281.1 million for the year 2000. To see how different the two models are, you should compare the graph for the uninhibited growth model, shown in Figure 6.1.3, with the graph of the inhibited growth model, shown in Figure 6.3.2.

Problems

1. Evaluate the following limits.

   (a) $\lim_{n \to \infty} \left(1 + \dfrac{1}{n}\right)^n$

   (b) $\lim_{n \to \infty} \left(1 + \dfrac{5}{n}\right)^n$

   (c) $\lim_{n \to \infty} \left(1 - \dfrac{2}{n}\right)^n$

   (d) $\lim_{n \to \infty} \left(1 - \dfrac{3}{n}\right)^n$

   (e) $\lim_{n \to \infty} \left(1 + \dfrac{2}{n^2}\right)^n$

   (f) $\lim_{n \to \infty} \left(1 - \dfrac{4}{n} + \dfrac{1}{n^2}\right)^n$

2. Suppose \$1500 is deposited in a bank account paying 5.5% interest. Find the amount in the account after 5 years if the interest is compounded (a) quarterly, (b) monthly, (c) weekly, (d) daily, and (e) continuously.

3. Suppose \$4500 is deposited in a bank account paying 6.25% interest. Find the amount in the account after 7 years if the interest is compounded (a) quarterly, (b) monthly, (c) weekly, (d) daily, and (e) continuously.

4. A customer deposits P dollars in a bank account. Which is more advantageous to the bank customer: 5% interest compounded continuously, 5.25% interest compounded monthly, or 5.5% interest compounded quarterly?

5. Let $A(x)$ be amount in a bank account after one year if \$1000 is deposited at 5% interest compounded x times per year.

   (a) Plot $A(x)$ on the interval $[1, 100]$.

   (b) Show that $A(x)$ is an increasing function on $(1, \infty)$.

6. A bone fossil is determined to have 5% of its original carbon-14 remaining. How old is the fossil?

7. Suppose an analysis of a bone fossil shows that it has between 4% and 6% percent of its original carbon-14. Find upper and lower bounds for the age of the fossil.

8. Carbon-11 has a half-life of 20 minutes. Given an initial amount x_0, find $x(t)$, the amount of carbon-11 remaining after t minutes. How long will it take before there is only 10% left? How long until only 5% remains?

9. Plutonium-239, the fuel for nuclear reactors, has a half-life of 24,000 years. Given an initial amount x_0, find $x(t)$, the amount of plutonium-239 remaining after t years. How

many years will it take before there is only 10% left? How many years until only 5% remains?

10. If 1% of a certain radioactive element decays in one year, what is the half-life of the element?

11. (a) In 1960 the population of the United States was 179.3 million and in 1970 it was 203.3 million. If $y(t)$ represents the size of the population of the United States t years after 1960, find an expression for $y(t)$ using an uninhibited growth model.

   (b) Use $y(t)$ from part (a) to predict the population of the United States in 1980, 1990, and 2000. How accurate are these predictions?

   (c) Let $x(t)$ be the population of the United States t years after 1960 as given by the inhibited growth model used in the last example in the section. Compare $y(t)$ to $x(t)$ by graphing them together over the interval $[0, 200]$.

12. The population of the United States was 3,929,214 in 1790, 5,308,483 in 1800, and 7,239,881 in 1810.

   (a) Let $y(t)$ be the population of the United States t years after 1790 as predicted by an uninhibited growth model using the data from 1790 and 1800. Graph $y(t)$ over the interval $[0, 100]$ and find the predicted population for 1810, 1820, 1840, 1870, 1900, and 1990. How accurate are these predictions?

   (b) Let $x(t)$ be the population of the United States t years after 1790 as predicted by an inhibited growth model using the data from 1790, 1800, and 1810. Graph $x(t)$ over the interval $[0, 200]$ and find the predicted population for 1820, 1840, 1870, 1900, and 1990. How accurate are these predictions? How do they compare with your results in a part (a)? What does this model predict for the eventual limiting population of the United States?

13. The population of the United States was 75,994,575 in 1900, 91,972,266 in 1910, and 105,710,620 in 1920.

   (a) Let $y(t)$ be the population of the United States t years after 1900 as predicted by an uninhibited growth model using the data from 1900 and 1910. Graph $y(t)$ over the interval $[0, 100]$ and find the predicted population for 1920, 1930, 1950, 1970, 1990, and 2000. How accurate are these predictions? Using this model, in what year will the population be twice what it was in 1900?

   (b) Let $x(t)$ be the population of the United States t years after 1900 as predicted by an inhibited growth model using the data from 1900, 1910, and 1920. Graph $x(t)$ over the interval $[0, 200]$ and find the predicted population for 1930, 1950, 1970, 1990, and 2000. How accurate are these predictions? How do they compare with your results in a part (a)? What does this model predict for the eventual limiting population of the United States? Using this model, in what year will the population be twice what it was in 1900?

14. Show that the graph of a solution to the logistic differential equation

$$\dot{x}(t) = \frac{\alpha}{M}x(t)(M - x(t)),$$

with $0 < x(0) < M$, is concave up when

$$x(t) > \frac{M}{2}$$

and concave down when

$$x(t) > \frac{M}{2}.$$

What does this say about the rate of growth of the population?