Difference Equations to Differential Equations

Section 6.7

Hyperbolic Functions

The final class of functions we will consider are the hyperbolic functions. In a sense these functions are not new to us since they may all be expressed in terms of the exponential function and its inverse, the natural logarithm function. However, we will see that they have many interesting and useful properties.

Definition For any real number x, the *hyperbolic sine* of x, denoted $\sinh(x)$, is defined by

$$\sinh(x) = \frac{1}{2}(e^x - e^{-x}) \tag{6.7.1}$$

and the *hyperbolic cosine* of x, denoted $\cosh(x)$, is defined by

$$\cosh(x) = \frac{1}{2}(e^x + e^{-x}). \tag{6.7.2}$$

Note that, for any real number t,

$$\begin{aligned}\cosh^2(t) - \sinh^2(t) &= \frac{1}{4}(e^t + e^{-t})^2 - \frac{1}{4}(e^t - e^{-t})^2 \\ &= \frac{1}{4}(e^{2t} + 2e^te^{-t} + e^{-2t}) - \frac{1}{4}(e^{2t} - 2e^te^{-t} + e^{-2t}) \\ &= \frac{1}{4}(2+2) \\ &= 1.\end{aligned}$$

Thus we have the useful identity

$$\cosh^2(t) - \sinh^2(t) = 1 \tag{6.7.3}$$

for any real number t. Put another way, $(\cosh(t), \sinh(t))$ is a point on the hyperbola $x^2 - y^2 = 1$. Hence we see an analogy between the hyperbolic cosine and sine functions and the cosine and sine functions: Whereas $(\cos(t), \sin(t))$ is a point on the circle $x^2 + y^2 = 1$, $(\cosh(t), \sinh(t))$ is a point on the hyperbola $x^2 - y^2 = 1$. In fact, the cosine and sine functions are sometimes referred to as the *circular* cosine and sine functions. We shall see many more similarities between the hyperbolic trigonometric functions and their circular counterparts as we proceed with our discussion.

To understand the graphs of the hyperbolic sine and cosine functions, we first note that, for any value of x,

$$\sinh(-x) = \frac{1}{2}(e^{-x} - e^x) = -\sinh(x), \tag{6.7.4}$$

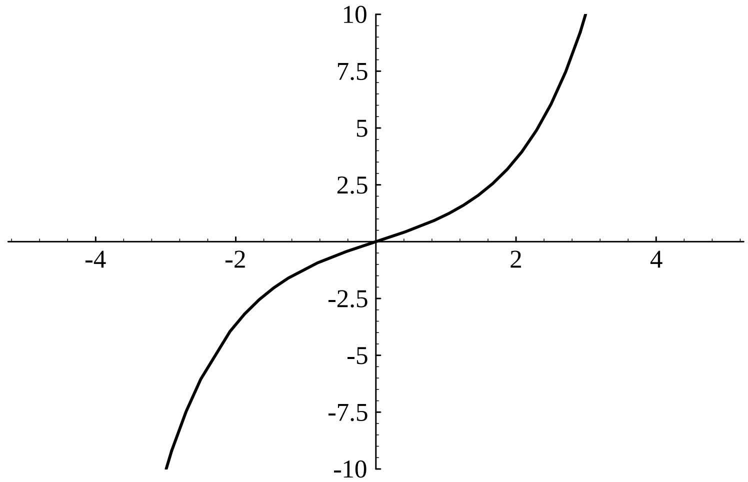

Figure 6.7.1 Graph of $y = \sinh(x)$

and

$$\cosh(-x) = \frac{1}{2}(e^{-x} + e^{x}) = \cosh(x). \tag{6.7.5}$$

Now for large values of x, $e^{-x} \approx 0$, in which case

$$\sinh(x) = \frac{1}{2}(e^{x} - e^{-x}) \approx = \frac{1}{2}e^{x}$$

and

$$\sinh(-x) = -\sinh(x) \approx -\frac{1}{2}e^{x}.$$

Thus the graph of $y = \sinh(x)$ appears as in Figure 6.7.1. Similarly, for large values of x,

$$\cosh(x) = \frac{1}{2}(e^{x} + e^{-x}) \approx \frac{1}{2}e^{x}$$

and

$$\cosh(-x) = \cosh(x) \approx \frac{1}{2}e^{x}.$$

The graph of $y = \cosh(x)$ is shown in Figure 6.7.2.

The derivatives of the hyperbolic sine and cosine functions follow immediately from their definitions. Namely,

$$\frac{d}{dx}\sinh(x) = \frac{d}{dx}\frac{1}{2}(e^{x} - e^{-x}) = \frac{1}{2}(e^{x} + e^{-x}) = \cosh(x)$$

and

$$\frac{d}{dx}\cosh(x) = \frac{d}{dx}\frac{1}{2}(e^{x} + e^{-x}) = \frac{1}{2}(e^{x} - e^{-x}) = \sinh(x).$$

Here again we see similarities between the circular and hyperbolic sine and cosine functions.

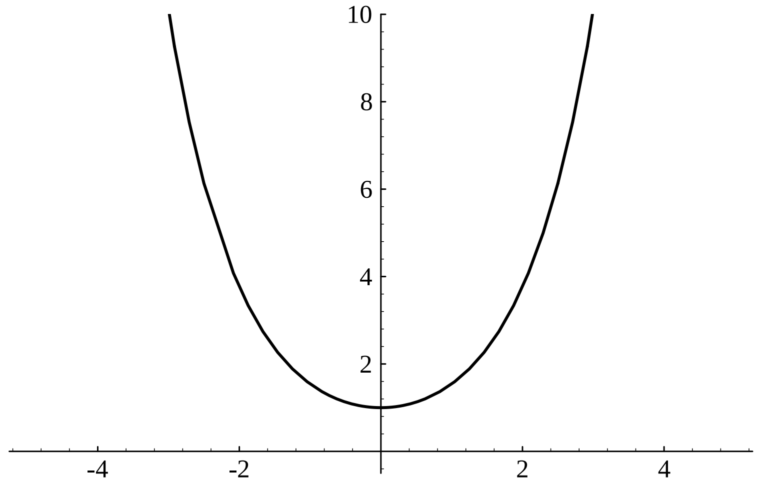

Figure 6.7.2 Graph of $y = \cosh(x)$

Proposition

$$\frac{d}{dx}\sinh(x) = \cosh(x). \tag{6.7.6}$$

$$\frac{d}{dx}\cosh(x) = \sinh(x). \tag{6.7.7}$$

As a consequence of this proposition, we also have

$$\int \sinh(x)dx = \cosh(x) + c \tag{6.7.8}$$

and

$$\int \cosh(x)dx = \sinh(x) + c. \tag{6.7.9}$$

Example Using the chain rule, we have

$$\frac{d}{dx}\sinh^2(3x) = 2\sinh(3x)\frac{d}{dx}\sinh(3x) = 6\sinh(3x)\cosh(3x).$$

Example Using the chain and product rules, we have

$$\begin{aligned}\frac{d}{dx}\sinh(2x)\cosh(2x) &= \sinh(2x)(2\sinh(2x)) + \cosh(2x)(2\cosh(2x)) \\ &= 2\sinh^2(2x) + 2\cosh^2(2x).\end{aligned}$$

Example Analogous to

$$\int \sin(3x)dx = -\frac{1}{3}\cos(x) + c,$$

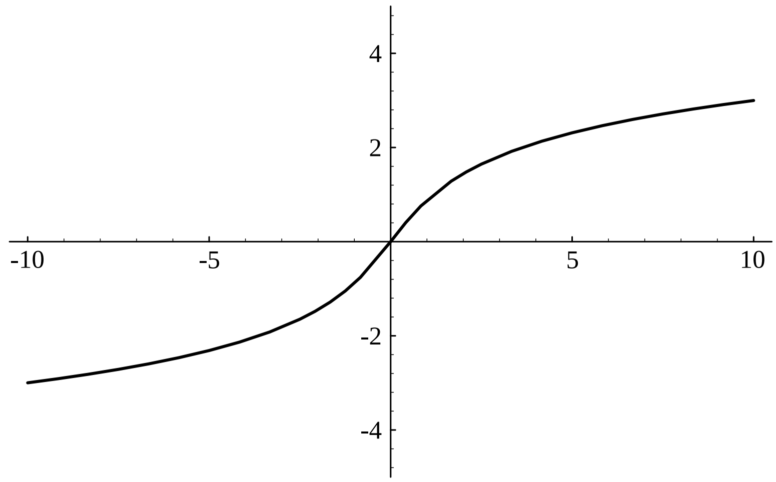

Figure 6.7.3 Graph of $y = \sinh^{-1}(x)$

we have

$$\int \sinh(3x)dx = \frac{1}{3}\cosh(3x) + c.$$

Example It is tempting to evaluate

$$\int e^{-x}\sinh(x)dx$$

using integration by parts in the same manner that we would evaluate

$$\int e^{-x}\sin(x)dx.$$

However, this integral is much easier if we notice that

$$e^{-x}\sinh(x) = e^{-x}\left(\frac{1}{2}(e^{x} - e^{-x})\right) = \frac{1}{2}(1 - e^{-2x}).$$

Hence

$$\int e^{-x}\sinh(x)dx = \frac{1}{2}\int(1 - e^{-2x})dx = \frac{x}{2} + \frac{1}{4}e^{-2x} + c.$$

Since $\frac{d}{dx}\sinh(x) = \cosh(x) > 0$ for all x, the hyperbolic sine function is increasing on the interval $(-\infty, \infty)$. Thus it has an inverse function, called the *inverse hyperbolic sine function*, with value at x denoted by $\sinh^{-1}(x)$. Since the domain and range of the hyperbolic sine function are both $(-\infty, \infty)$, the domain and range of the inverse hyperbolic sine function are also both $(-\infty, \infty)$. As usual with inverse functions,

$$y = \sinh^{-1}(x) \text{ if and only if } \sinh(y) = x. \tag{6.7.10}$$

The graph of $y = \sinh^{-1}(x)$ is shown in Figure 6.7.3.

Example The hyperbolic sine function and its inverse provide an alternative method for evaluating

$$\int \frac{1}{\sqrt{1+x^2}}\, dx.$$

Namely, if we make the substitution

$$\begin{aligned} x &= \sinh(u), -\infty < u < \infty, \\ dx &= \cosh(u)du, \end{aligned}$$

then

$$\sqrt{1+x^2} = \sqrt{1+\sinh^2(u)} = \sqrt{\cosh^2(u)} = \cosh(u),$$

where the second equality follows from the identity $\cosh^2(u) - \sinh^2(u) = 1$ and the last equality from the fact that $\cosh(u) > 0$ for all u. Hence

$$\int \frac{1}{\sqrt{1+x^2}}\, dx = \int \frac{\cosh(u)}{\cosh(u)}\, du = \int du = u + c = \sinh^{-1}(x) + c.$$

The following proposition is a consequence of the previous example.

Proposition

$$\frac{d}{dx}\sinh^{-1}(x) = \frac{1}{\sqrt{1+x^2}}. \qquad (6.7.11)$$

In Section 6.6 we saw, using the substitution $x = \tan(u)$, $-\frac{\pi}{2} < u < \frac{\pi}{2}$, that

$$\int \frac{1}{\sqrt{1+x^2}}\, dx = \log\left|x + \sqrt{1+x^2}\right| + c.$$

Since two antiderivatives of a function can differ at most by a constant, there must exist a constant k such that

$$\sinh^{-1}(x) = \log\left|x + \sqrt{1+x^2}\right| + k$$

for all x. Evaluating both sides of this equality at $x = 0$, we have

$$0 = \sinh^{-1}(0) = \log(1) + k = k.$$

Thus $k = 0$ and

$$\sinh^{-1}(x) = \log\left|x + \sqrt{1+x^2}\right| \qquad (6.7.12)$$

for all x. Since the hyperbolic sine function is defined in terms of the exponential function, we should not find it surprising that the inverse hyperbolic sine function may be expressed in terms of the natural logarithm function.

Similarly, since $\frac{d}{dx}\cosh(x) = \sinh(x) > 0$ for all $x > 0$, the hyperbolic cosine function is increasing on the interval $[0, \infty)$, and so has an inverse if we restrict its domain to $[0, \infty)$. That is, we define the *inverse hyperbolic cosine* function by the relationship

$$y = \cosh^{-1}(x) \text{ if and only if } x = \cosh(y), \qquad (6.7.13)$$

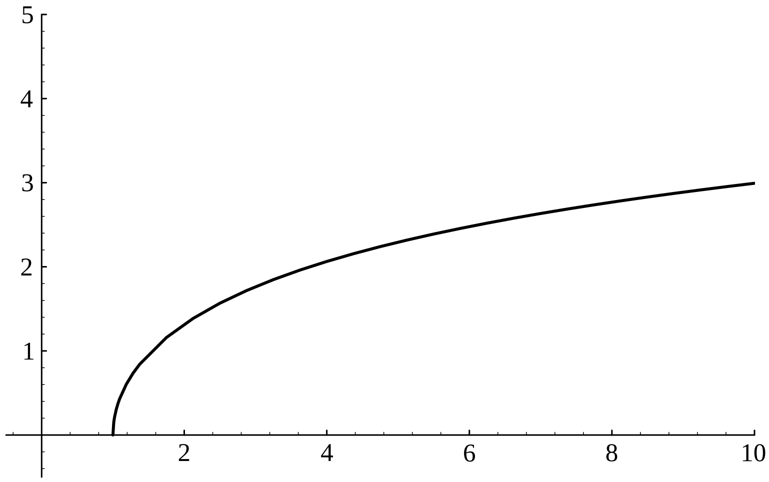

Figure 6.7.4 Graph of $y = \cosh^{-1}(x)$

where we require $y \geq 0$. Note that since $\cosh(x) \geq 1$ for all x, the domain of of the inverse hyperbolic cosine function is $[1, \infty)$. The graph of $y = \cosh^{-1}(x)$ is shown in Figure 6.7.4.

In Problem 3 at the end of this section you are asked to show that

$$\int \frac{1}{\sqrt{x^2 - 1}}\, dx = \cosh^{-1}(x) + c$$

for $x > 1$, from which the following proposition follows.

Proposition

$$\frac{d}{dx} \cosh^{-1}(x) = \frac{1}{\sqrt{x^2 - 1}}. \tag{6.7.14}$$

In the same problem you are asked to show that, for $x \geq 1$,

$$\cosh^{-1}(x) = \log \left| x + \sqrt{x^2 - 1} \right| \tag{6.7.15}$$

Example In Section 6.6 we evaluated the integral

$$\int \frac{1}{\sqrt{x^2 - 9}}\, dx,$$

for $x > 3$, using the substitution $x = 3\sec(u)$, $0 < u < \frac{\pi}{2}$. The substitution

$$\begin{aligned} x &= 3\cosh(u), u > 0, \\ dx &= 3\sinh(u) du \end{aligned}$$

provides a somewhat simpler approach. Namely,

$$\begin{aligned}
\int \frac{1}{\sqrt{x^2-9}}\,dx &= \int \frac{3\sinh(u)}{\sqrt{9\cosh^2(u)-9}}\,du \\
&= \int \frac{3\sinh(u)}{3\sqrt{\cosh^2(u)-1}}\,du \\
&= \int \frac{\sinh(u)}{\sqrt{\sinh^2(u)}}\,du \\
&= \int \frac{\sinh(u)}{\sinh(u)}\,du \\
&= \int du \\
&= u + c \\
&= \cosh^{-1}\left(\frac{x}{3}\right) + c,
\end{aligned}$$

where we have used the fact that $\sinh(u) > 0$ when $u > 0$.

Having defined the hyperbolic sine and cosine functions, it is possible to define four more hyperbolic trigonometric functions in analogy with the circular trigonometric functions. Namely, the *hyperbolic tangent function* is given by

$$\tanh(x) = \frac{\sinh(x)}{\cosh(x)}, \tag{6.7.16}$$

where $-\infty < x < \infty$; the *hyperbolic cotangent function* by

$$\coth(x) = \frac{\cosh(x)}{\sinh(x)}, \tag{6.7.17}$$

where $x \neq 0$; the *hyperbolic secant function* by

$$\operatorname{sech}(x) = \frac{1}{\cosh(x)}, \tag{6.7.18}$$

where $-\infty < x < \infty$; and the *hyperbolic cosecant function* by

$$\operatorname{csch}(x) = \frac{1}{\sinh(x)}, \tag{6.7.19}$$

where $x \neq 0$. In Problem 5 at the end of this section you are asked to verify the following results.

Proposition

$$\frac{d}{dx}\tanh(x) = \operatorname{sech}^2(x). \tag{6.7.20}$$

Proposition

$$\frac{d}{dx}\coth(x) = -\text{csch}^2(x) \tag{6.7.21}$$

Proposition

$$\frac{d}{dx}\text{sech}(x) = -\text{sech}(x)\tanh(x). \tag{6.7.22}$$

Proposition

$$\frac{d}{dx}\text{csch}(x) = -\text{csch}(x)\coth(x). \tag{6.7.23}$$

Since

$$\tanh(x) = \frac{\sinh(x)}{\cosh(x)} = \frac{e^x - e^{-x}}{e^x + e^{-x}},$$

we have

$$\begin{aligned}\lim_{x\to\infty}\tanh(x) &= \lim_{x\to\infty}\frac{e^x - e^{-x}}{e^x + e^{-x}}\\ &= \lim_{x\to\infty}\frac{e^x(1-e^{-2x})}{e^x(1+e^{-2x})}\\ &= \lim_{x\to\infty}\frac{1-e^{-2x}}{1+e^{-2x}}\\ &= 1\end{aligned}$$

and

$$\begin{aligned}\lim_{x\to-\infty}\tanh(x) &= \lim_{x\to-\infty}\frac{e^x - e^{-x}}{e^x + e^{-x}}\\ &= \lim_{x\to-\infty}\frac{e^{-x}(e^{2x}-1)}{e^{-x}(e^{2x}+1)}\\ &= \lim_{x\to-\infty}\frac{e^{2x}-1}{e^{2x}+1}\\ &= -1.\end{aligned}$$

Hence $y = 1$ and $y = -1$ are both horizontal asymptotes for the graph of $y = \tanh(x)$. Combining this information with $\tanh(0) = 0$ and

$$\frac{d}{dx}\tanh(x) = \text{sech}^2(x) > 0$$

for all x, we can see why the graph of $y = \tanh(x)$ looks as it does in Figure 6.7.5.

Since the hyperbolic tangent function is increasing on $(-\infty, \infty)$, it has an inverse, called the *inverse hyperbolic tangent function*, with value at x denoted by $\tanh^{-1}(x)$. That is, as usual,

$$y = \tanh^{-1}(x) \text{ if and only if } \tanh(y) = x. \tag{6.7.24}$$

The domain of the inverse hyperbolic tangent function is $(-1, 1)$ the range of the hyperbolic tangent function, and its range is $(-\infty, \infty)$, the domain of the hyperbolic tangent

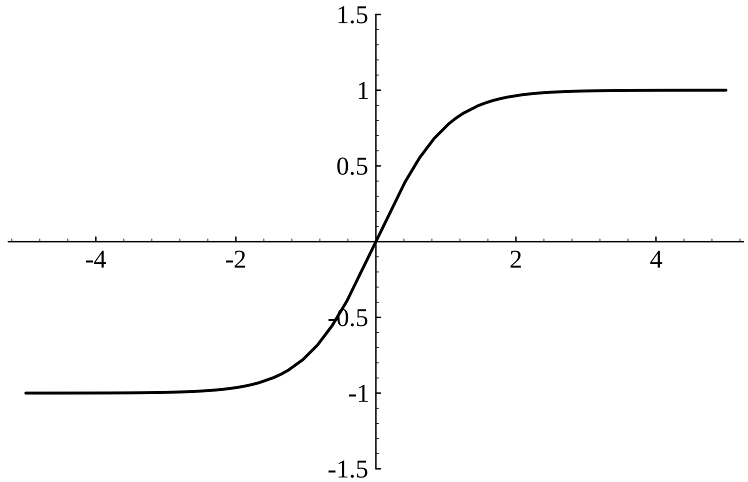

Figure 6.7.5 Graph of $y = \tanh(x)$

function. Corresponding to the horizontal asymptotes of the graph of the hyperbolic tangent function, the graph of the inverse hyperbolic tangent function has vertical asymptotes $x = -1$ and $x = 1$, as shown in Figure 6.7.6.

Example As an alternative to using partial fractions, we may evaluate the integral

$$\int \frac{1}{1-x^2}\, dx$$

for $-1 < x < 1$ using the substitution

$$x = \tanh(u), -\infty < u < \infty,$$
$$dx = \operatorname{sech}^2(u)du.$$

Then

$$\int \frac{1}{1-x^2}\, dx = \int \frac{\operatorname{sech}^2(u)}{1-\tanh^2(u)}\, du.$$

Now from the identity

$$\cosh^2(x) - \sinh^2(x) = 1$$

we obtain

$$\frac{\cosh^2(x)}{\cosh^2(x)} - \frac{\sinh^2(x)}{\cosh^2(x)} = \frac{1}{\cosh^2(x)}.$$

In other words,

$$1 - \tanh^2(x) = \operatorname{sech}^2(x). \tag{6.7.25}$$

Hence

$$\int \frac{1}{1-x^2}\, dx = \int \frac{\operatorname{sech}^2(u)}{\operatorname{sech}^2(u)}\, du = \int du = u + c = \tanh^{-1}(x) + c.$$

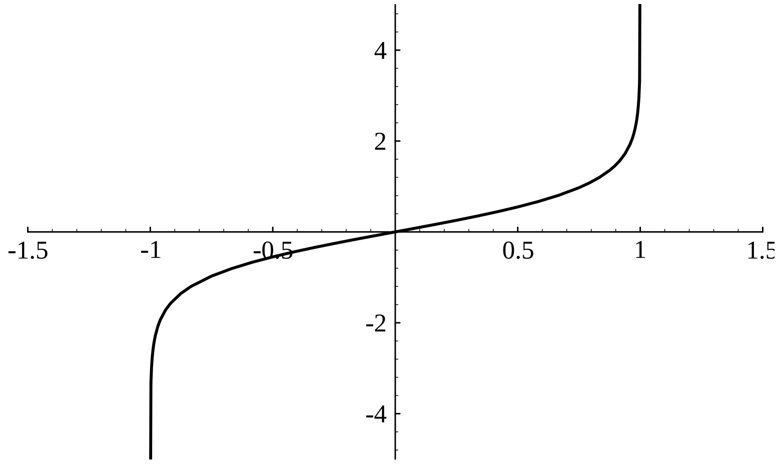

Figure 6.7.6 Graph of $y = \tanh^{-1}(x)$

Note that (6.7.25) gives us the useful identity

$$\tanh^2(x) + \operatorname{sech}^2(x) = 1 \tag{6.7.26}$$

for all x. Moreover, we have the following proposition as a consequence of this example.

Proposition

$$\frac{d}{dx}\tanh^{-1}(x) = \frac{1}{1-x^2}. \tag{6.7.27}$$

If we were to use partial fractions to evaluate the integral of the previous example, we would obtain, for $-1 < x < 1$,

$$\int \frac{1}{1-x^2}\,dx = \frac{1}{2}\log(1-x) - \frac{1}{2}\log(1+x) + c = \frac{1}{2}\log\left(\frac{1+x}{1-x}\right) + c.$$

It follows that

$$\tanh^{-1}(x) = \frac{1}{2}\log\left(\frac{1+x}{1-x}\right) + k$$

for some constant k. Evaluating at 0, we have

$$0 = 0 + k.$$

Thus $k = 0$ and we have

$$\tanh^{-1}(x) = \frac{1}{2}\log\left(\frac{1+x}{1-x}\right) \tag{6.7.28}$$

for $1 < x < 1$.

Problems

1. Differentiate each of the following functions.

   (a) $f(x) = \sinh(3x)$
   (b) $g(t) = 3t\cosh(4t)$
   (c) $f(t) = 3t\sinh(t)\cosh(2t)$
   (d) $g(x) = 4x\sinh(3x^2 - 1)$
   (e) $y(t) = 5t^2\cosh^2(4t)$
   (f) $f(t) = 3\cosh^2(2t) - 13\sinh(3t^2)$

2. Evaluate each of the following integrals.

   (a) $\int \sinh(3x)dx$
   (b) $\int \cosh(4t - 3)dt$
   (c) $\int \sinh(z)\cosh(z)dz$
   (d) $\int 3x\sinh(2x)dx$
   (e) $\int e^{-2t}\cosh(2t)dt$
   (f) $\int \cosh^2(x)\sinh(x)dx$
   (g) $\int 5t^2\cosh(2t)dt$
   (h) $\int \frac{\sinh(t)}{\cosh^2(t)}\,dt$

3. (a) Use the substitution $x = \cosh(u), u > 0$, to show that

   $$\int \frac{1}{\sqrt{x^2 - 1}}\,dx = \cosh^{-1}(x) + c$$

   for $x > 1$.

   (b) Use the substitution $x = \sec(u), 0 < u < \frac{\pi}{2}$, to show that

   $$\int \frac{1}{\sqrt{x^2 - 1}}\,dx = \log\left|x + \sqrt{x^2 - 1}\right| + c$$

   for $x > 1$.

   (c) Using (a) and (b), show that

   $$\cosh^{-1}(x) = \log\left|x + \sqrt{x^2 - 1}\right|$$

   for $x > 1$.

4. Evaluate the following integrals.

   (a) $\int \frac{1}{\sqrt{4 + x^2}}\,dx$
   (b) $\int \frac{1}{\sqrt{x^2 - 4}}\,dx, x > 2$
   (c) $\int \frac{3}{\sqrt{9 + 3t^2}}\,dt$
   (d) $\int \frac{1}{\sqrt{x^2 - 1}}\,dx, x < -1$

5. Verify the following derivatives.

   (a) $\frac{d}{dx}\tanh(x) = \operatorname{sech}^2(x)$
   (b) $\frac{d}{dx}\coth(x) = -\operatorname{csch}^2(x)$

(c) $\dfrac{d}{dx}\text{sech}(x) = -\text{sech}(x)\tanh(x)$

(d) $\dfrac{d}{dx}\text{csch}(x) = -\text{csch}(x)\coth(x)$

6. Differentiate each of the following functions.

   (a) $f(x) = 3x\tanh(4x)$

   (b) $g(t) = \text{sech}^2(3t)$

   (c) $h(\theta) = 4\tanh^2(\theta)\text{sech}(\theta)$

   (d) $f(x) = 5x\text{sech}(4x) - 21\tanh^3(4x)$

7. Evaluate each of the following integrals.

   (a) $\displaystyle\int \tanh(x)dx$

   (b) $\displaystyle\int \tanh(2x)\text{sech}(2x)dx$

   (c) $\displaystyle\int \frac{1}{4-x^2}\,dx$

   (d) $\displaystyle\int \frac{5}{9-3t^2}\,dt$

8. Graph each of the following functions on an appropriate interval.

   (a) $y = \text{sech}(x)$

   (b) $y = \coth(x)$

   (c) $y = \text{csch}(x)$

   (d) $y = 3\tanh(4x)$