Lecture 16: Implicit Differentiation

16.1 Implicit differentiation

Example Let C be the circle with equation $x^2 + y^2 = 25$. Note that C is not the graph of a function. Nevertheless, the curve has a tangent line at every point (a, b) with $b \neq 0$. Moreover, every point (a, b) on C with $b \neq 0$ lies on a piece of the circle which is the graph of some function, namely, $y = -\sqrt{25 - x^2}$ if $b < 0$ or $y = \sqrt{25 - x^2}$ if $b > 0$. Hence we may treat y as a function of x for some interval containing a. Now

$$\begin{aligned} x^2 + y^2 = 25 &\Rightarrow \frac{d}{dx}(x^2 + y^2) = \frac{d}{dx}(25) \\ &\Rightarrow 2x + 2y\frac{dy}{dx} = 0 \\ &\Rightarrow \frac{dy}{dx} = -\frac{x}{y}, y \neq 0. \end{aligned}$$

For example, at $(3, 4)$,

$$\left.\frac{dy}{dx}\right|_{(x,y)=(3,4)} = -\frac{3}{4}.$$

Hence the equation of the line tangent to C at $(3, 4)$ is

$$y = -\frac{3}{4}(x - 3) + 4.$$

In general, to find the slope of the line tangent to the curve with equation $g(x, y) = c$, compute $\frac{d}{dx}g(x, y) = \frac{d}{dx}(c)$ and solve for $\frac{dy}{dx}$.

Example To find the slope of the line tangent to the curve

$$y^5 - 3xy^2 + 3x^2 = 7$$

at $(2, 1)$, we have

$$\begin{aligned} \frac{d}{dx}(y^5 - 3xy^2 + 3x^2) = \frac{d}{dx}(7) &\Rightarrow 5y^4\frac{dy}{dx} - 3x(2y\frac{dy}{dx}) - 3y^2 + 6x = 0 \\ &\Rightarrow \frac{dy}{dx} = \frac{3y^2 - 6x}{5y^4 - 6xy} \\ &\Rightarrow \left.\frac{dy}{dx}\right|_{(x,y)=(2,1)} = \frac{3 - 12}{5 - 12} = \frac{9}{7}. \end{aligned}$$

Hence the equation of the line tangent to the curve at $(2, 1)$ is

$$y = \frac{9}{7}(x - 2) + 1.$$

The following plot shows the curve and the tangent line plotted together.