1 Some Important Limits

We have seen the definition of sequences and the limit of a sequence. When you studied limits of functions in Calculus I, you may recall that you preferred to use limit rules and other known limits to calculate limits of new functions rather than to use the definition of limit. The same holds true for limits of sequences. The purpose of this ection is to calculate some fairly tricky limits so that we can use them later in finding other limits. All ofthe proofs are in the book, but I am adding a few notes here in an attempt to clarify things a bit.

- Let $x > 0$. *Then* $\lim_{n\to\infty} x^{1/n} = 1$

  For any $x > 0$, $\ln x^{1/n} = \frac{1}{n}\ln x$. Since x is fixed, this goes to 0 as $n \to \infty$, i.e. $\lim_{n\to\infty} \frac{1}{n}\ln x = 0$. Since e^x is continuous at $x = 0$, we have

  $$e^{\lim_{n\to\infty}\frac{1}{n}\ln n} = \lim_{n\to\infty} e^{\frac{1}{n}\ln n} = \lim_{n\to\infty} x^{1/n}$$

  But,

  $$e^{\lim_{n\to\infty}\frac{1}{n}\ln n} = e^0 = 1$$

  and the result follows.

  $$1 = e^0 = e^{\lim_{n\to\infty}\frac{1}{n}\ln x} = \lim_{n\to\infty} e^{\frac{1}{n}\ln x} = \lim_{n\to\infty} x^{1/n}$$

- If $|x| < 1$, then $\lim_{n\to\infty} x^n = 0$

- If $\alpha > 0$, $\dfrac{1}{n^\alpha} \to 0$ as $n \to \infty$

- For each real x, $\lim_{n\to\infty} \dfrac{x^n}{n!} = 0$

  The limit indicates that even with a huge x like 1,000,000, that the denominator will eventually dominate the numerator and force the quotient to become small as n increases. To see why, we notice that as we go from the nth term to the $(n+1)$st term that we multiply in the numerator by x and in the denominator by $(n+1)$. So, even if x is 1,000,000, by the time we get out to $n = 1,000,001$ we are now multiplying in the denominator by a number larger than we are multipliyng in the numerator. As the disparity increases as n increases, it seems plausible that this will force the quotient to zero in the limit. We use this idea to o demonstrate this for any x. We fix an x once and for all, and choose a positive integer k, also fixed, which has the property that $k > |x|$. When $n > k+1$ we write

$$\frac{k^n}{n!} = \frac{k^k}{k!}\left[\frac{k}{k+1}\,\frac{k}{k+2}\,\frac{k}{k+1}\cdots\frac{k}{n-2}\,\frac{k}{n-1}\right]\left(\frac{k}{n}\right) < \frac{k^{k+1}}{k!}\cdot\frac{1}{n}$$

Since $n > k+1$, we know that $n-1 > k$ and so the quantity above in the square brackets is less than 1. This gives the inequality above.

Now, since $k > |k|$,

$$0 < \frac{|x|^n}{n!} < \frac{k^n}{n!} < \left(\frac{k^{k+1}}{k!}\right)\frac{1}{n}$$

Observe that the quantity $\frac{k^{k+1}}{k!}$ is fixed. If we take limits, we get

$$\lim_{n\to\infty} 0 < \lim_{n\to\infty}\frac{|x|^n}{n!} < \lim_{n\to\infty}\frac{k^n}{n!} < \lim_{n\to\infty}\left(\frac{k^{k+1}}{k!}\right)\frac{1}{n} = \left(\frac{k^{k+1}}{k!}\right)\lim_{n\to\infty}\frac{1}{n} = 0$$

By the pinching theorem, we have

$$\lim_{n\to\infty}\frac{|x|^n}{n!} = 0$$

which implies the result.

- $\displaystyle\lim_{n\to\infty}\frac{\ln n}{n} = 0$
- $n^{1/n} \to 1$ as $n \to \infty$
- $\displaystyle\left(1+\frac{x}{n}\right)^n \to e^x$ as $n \to \infty$