Difference Equations
to
Differential Equations

Section 4.6

Improper Integrals

In this section we will make two extensions to our definition of the definite integral. The first will cover integrals of functions over intervals of the form $[a, \infty]$ and $(-\infty, b]$, where a and b are fixed real numbers, as well as the interval $(-\infty, \infty)$, while the second will cover integrals of functions which have infinite discontinuities. An integral of either one of these two types is called an *improper integral*.

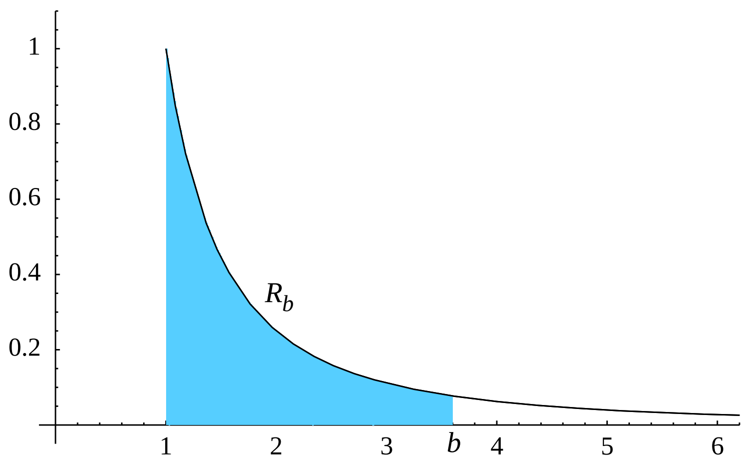

Figure 4.6.1 Area of R_b approaches area under $y = \dfrac{1}{x^2}$ as b increases

First, consider a function f defined on an interval $[a, \infty)$ with the property that f is integrable on every interval $[a, b]$ with $a < b < \infty$. For example, the function

$$f(x) = \frac{1}{x^2}$$

is defined for all x in $[1, \infty)$ and, since it is continuous on $[1, \infty)$, is integrable on any interval $[1, b]$ with $1 < b < \infty$. If we let R_b be the region beneath the graph of f over the interval $[1, b]$ and we let R be the region beneath the graph of f over the interval $[1, \infty)$, then we would expect that the area of R_b would approach the area of R in the limit as b goes to infinity (see Figure 4.6.1). In terms of integrals, this is saying that it would seem reasonable to define

$$\int_1^\infty \frac{1}{x^2}\,dx = \lim_{b\to\infty} \int_1^b \frac{1}{x^2}\,dx.$$

That is, we should have

$$\begin{aligned}\int_1^\infty \frac{1}{x^2}\,dx &= \lim_{b\to\infty}\int_1^b \frac{1}{x^2}\,dx\\ &= \lim_{b\to\infty} -\frac{1}{x}\Big|_1^b\\ &= \lim_{b\to\infty}\left(-\frac{1}{b}+1\right)\\ &= 1.\end{aligned}$$

Geometrically, this result says that R has finite area, namely, 1, even though it has infinite length.

We now state a general definition for this type of integral.

Definition If f is defined on $[a,\infty)$ and integrable on $[a,b]$ for all $a<b<\infty$, then we define

$$\int_a^\infty f(x)dx = \lim_{b\to\infty}\int_a^b f(x)dx, \tag{4.6.1}$$

provided the limit exists. Similarly, if f is defined on $(-\infty,b]$ and integrable on $[a,b]$ for all $-\infty<a<b$, then we define

$$\int_{-\infty}^b f(x)dx = \lim_{a\to-\infty}\int_a^b f(x)dx, \tag{4.6.2}$$

provided the limit exists. Finally, if f is defined on $(-\infty,\infty)$ and integrable on any finite interval $[a,b]$, then we define

$$\int_{-\infty}^\infty f(x)dx = \int_{-\infty}^0 f(x)dx + \int_0^\infty f(x)dx, \tag{4.6.3}$$

provided both of the integrals on the right exist. In each case where the appropriate limit exists, we say the integral *converges*; otherwise, the integral is said to *diverge*.

Note that the use of 0 in (4.6.3) is not crucial; all that is important is that the integral is broken into two pieces, the meaning of each of the pieces already having been covered in the earlier parts of the definition.

Example The integral

$$\int_3^\infty \frac{1}{x^3}\,dx$$

converges, since

$$\begin{aligned}\int_3^\infty \frac{1}{x^3}\,dx &= \lim_{b\to\infty}\int_3^b \frac{1}{x^3}\,dx\\ &= \lim_{b\to\infty} -\frac{1}{2x^2}\Big|_3^b\end{aligned}$$

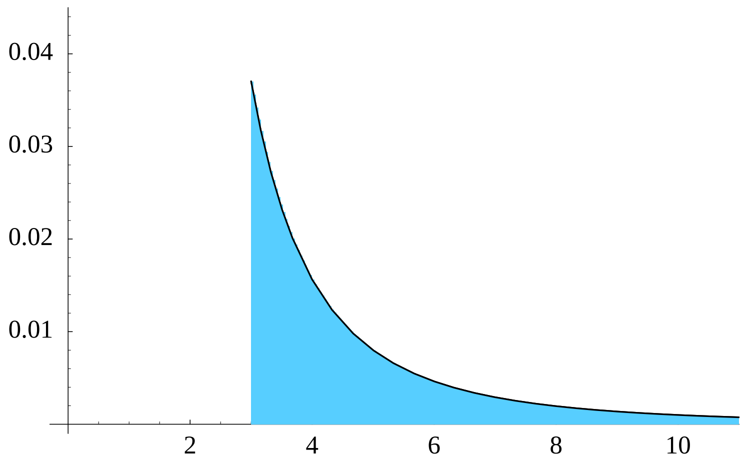

Figure 4.6.2 Region beneath $y = \dfrac{1}{x^3}$ beginning at 3

$$= \lim_{b \to \infty} \left(\frac{1}{2b^2} + \frac{1}{18} \right)$$
$$= \frac{1}{18}.$$

See Figure 4.6.2.

Example The integral

$$\int_2^\infty \frac{1}{\sqrt{x}}\, dx$$

diverges, since

$$\lim_{b \to \infty} \int_2^b \frac{1}{\sqrt{x}}\, dx = \lim_{b \to \infty} 2\sqrt{x}\, \Big|_2^b = \lim_{b \to \infty} (2\sqrt{b} - 2\sqrt{2}) = \infty.$$

See Figure 4.6.3.

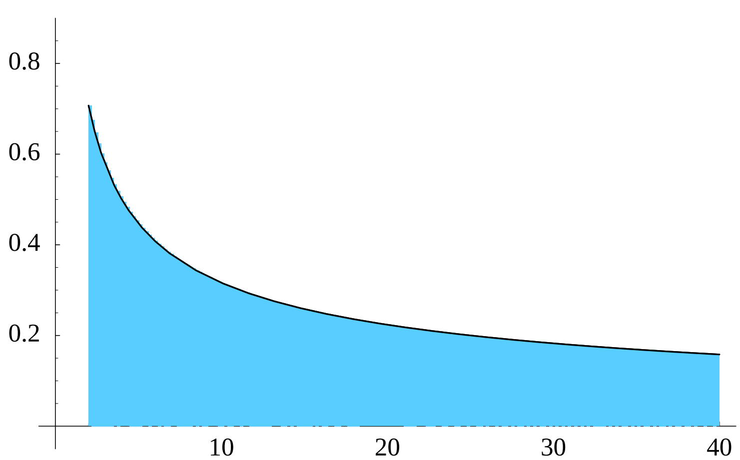

Figure 4.6.3 Region beneath $y = \dfrac{1}{\sqrt{x}}$ beginning at 2

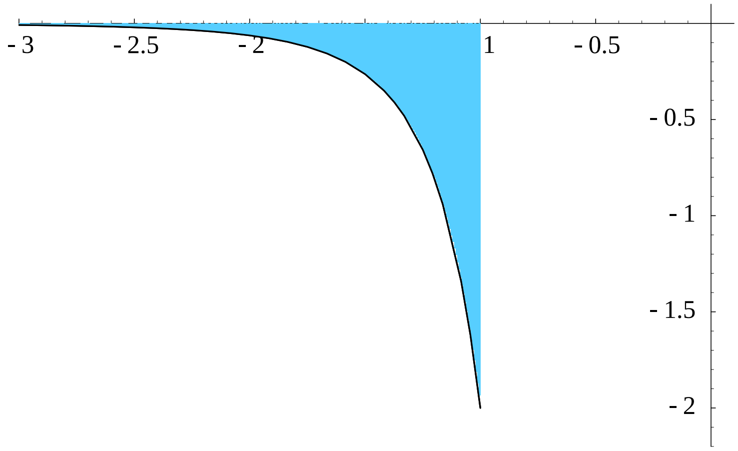

Figure 4.6.4 Region above $y = \dfrac{2}{x^5}$ ending at -1

Example The integral

$$\int_{-\infty}^{-1} \frac{2}{x^5}\, dx$$

converges, since

$$\begin{aligned}
\int_{-\infty}^{-1} \frac{2}{x^5}\, dx &= \lim_{a\to-\infty} \int_a^{-1} \frac{2}{x^5}\, dx \\
&= \lim_{a\to-\infty} -\frac{2}{4x^4}\bigg|_a^{-1} \\
&= \lim_{a\to-\infty} \left(-\frac{1}{2} + \frac{1}{2a^4}\right) \\
&= -\frac{1}{2}.
\end{aligned}$$

See Figure 4.6.4.

Example The integral

$$\int_{-\infty}^{\infty} \frac{x}{(1+x^2)^2}\, dx$$

converges, since

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{x}{(1+x^2)^2}\, dx &= \int_{-\infty}^{0} \frac{x}{(1+x^2)^2}\, dx + \int_0^{\infty} \frac{x}{(1+x^2)^2}\, dx \\
&= \lim_{a\to-\infty} \int_a^0 \frac{x}{(1+x^2)^2}\, dx + \lim_{b\to\infty} \int_0^b \frac{x}{(1+x^2)^2}\, dx \\
&= \lim_{a\to-\infty} -\frac{1}{2(1+x^2)}\bigg|_a^0 + \lim_{b\to\infty} -\frac{1}{2(1+x^2)}\bigg|_0^b \\
&= \lim_{a\to-\infty} \left(-\frac{1}{2} + \frac{1}{2(1+a^2)}\right) + \lim_{b\to\infty} \left(-\frac{1}{2(1+b^2)} + \frac{1}{2}\right)
\end{aligned}$$

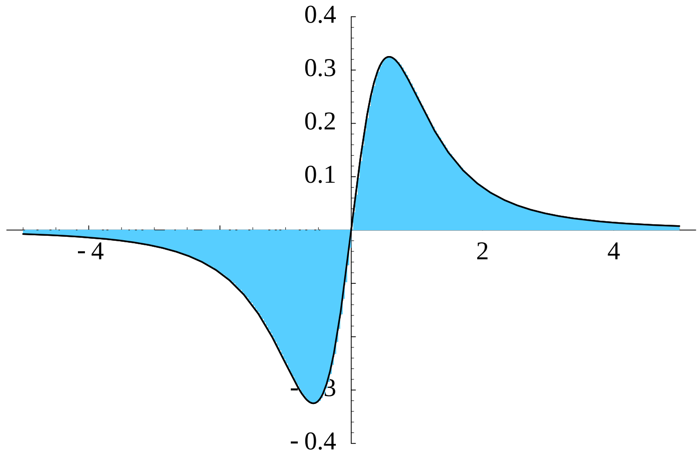

Figure 4.6.5 Region between $y = \dfrac{x}{(1+x^2)^2}$ and the x-axis

$$= -\frac{1}{2} + \frac{1}{2}$$
$$= 0.$$

Note that you could use the substitution $u = 1 + x^2$ to help evaluate the integral in this example. See Figure 4.6.5.

It is frequently important to know that an integral $\int_a^\infty f(x)dx$ converges even if we cannot compute its value exactly. For example, before trying to find numerical approximations for such an integral one should first check that it converges. We will first consider the following situation: Suppose f and g are defined on $[a, \infty)$, integrable on $[a, b]$ for all $a < b < \infty$, and $0 \leq f(x) \leq g(x)$ for all x in $[a, \infty)$. Moreover, suppose we know that $\int_a^\infty g(x)dx$ converges. Let

$$M = \int_a^\infty g(x)dx, \tag{4.6.4}$$

$$G(b) = \int_a^b g(x)dx, \tag{4.6.5}$$

and

$$F(b) = \int_a^b f(x)dx \tag{4.6.6}$$

for all $b \geq a$. Now for any $b \geq a$,

$$M = \int_a^\infty g(x)dx = \int_a^b g(x)dx + \int_b^\infty g(x)dx = G(b) + \int_b^\infty g(x)dx. \tag{4.6.7}$$

Since $g(x) \geq 0$ for all $x \geq a$,

$$\int_b^\infty g(x)dx \geq 0. \tag{4.6.8}$$

Thus (4.6.7) implies that

$$G(b) = M - \int_b^\infty g(x)dx \leq M \tag{4.6.9}$$

for all $b \geq a$. Moreover, $f(x) \leq g(x)$ for all $x \geq a$, so

$$F(b) = \int_a^b f(x)dx \leq \int_a^b g(x)dx = G(b) \tag{4.6.10}$$

for all $b \geq a$. Putting (4.6.9) and (4.6.10) together, we have $F(b) \leq M$ for all $b \geq a$. Furthermore, for any $c \geq b \geq a$,

$$F(c) = \int_a^c f(x)dx = \int_a^b f(x)dx + \int_b^c f(x)dx \geq \int_a^b f(x)dx = F(b), \tag{4.6.11}$$

where we know

$$\int_b^c f(x)dx \geq 0 \tag{4.6.12}$$

because $f(x) \geq 0$ for all $x \geq a$. From (4.6.11) we conclude that F is a nondecreasing function. Since we already know that F is bounded by M, it follows from our result about bounded nondecreasing sequences in Section 1.2 that

$$\lim_{b\to\infty} F(b) = \lim_{b\to\infty} \int_a^b f(x)dx \tag{4.6.13}$$

exists. That is,

$$\int_a^\infty f(x)dx = \lim_{b\to\infty} \int_a^b f(x)dx \tag{4.6.14}$$

converges. Moreover, since $F(b) \leq M$ for all $b \geq a$,

$$\int_a^\infty f(x)dx = \lim_{b\to\infty} F(b) \leq M = \int_a^\infty g(x)dx. \tag{4.6.15}$$

On the other hand, suppose f and g are defined on $[a, \infty)$, integrable on $[a, b]$ for all $a < b < \infty$, $0 \leq f(x) \leq g(x)$ for all x in $[a, \infty)$, and $\int_a^\infty f(x)dx$ diverges. If we define F and G as above, then $F(b)$ is nondecreasing and without a limit as b increases toward ∞. Hence it follows, again from our results in Section 1.2, that we must have

$$\lim_{b\to\infty} F(b) = \infty. \tag{4.6.16}$$

Since, as above, $G(b) \geq F(b)$ for all $b \geq a$, (4.6.16) implies that

$$\lim_{b\to\infty} \int_a^b g(x)dx = \lim_{b\to\infty} G(b) = \infty. \tag{4.6.17}$$

In particular, $\int_a^\infty g(x)dx$ diverges.

We summarize the previous results in the next proposition.

Proposition Suppose f and g are defined on $[a,\infty)$, integrable on $[a,b]$ for all $a<b<\infty$, and $0 \leq f(x) \leq g(x)$ for all x in $[a,\infty)$. If $\int_a^\infty g(x)dx$ converges, then $\int_a^\infty f(x)dx$ converges and
$$0 \leq \int_a^\infty f(x)dx \leq \int_a^\infty g(x)dx. \tag{4.6.18}$$
If $\int_a^\infty f(x)dx$ diverges, then $\int_a^\infty g(x)dx$ diverges.

Similar results hold for integrals on intervals of the form $(-\infty,b]$ and $(-\infty,\infty)$.

Example At present we cannot use the Fundamental Theorem to evaluate
$$\int_0^\infty \frac{1}{1+x^2}\,dx$$
because we do not know an antiderivative for
$$f(x) = \frac{1}{1+x^2}$$
(although we will find one in Section 6.5). However, since $x^2 < 1+x^2$ for all values of x, we know that
$$0 < \frac{1}{1+x^2} < \frac{1}{x^2}$$
for all $x>0$. Now we saw above that
$$\int_1^\infty \frac{1}{x^2}\,dx = 1,$$
so we know, by the previous proposition, that
$$\int_1^\infty \frac{1}{1+x^2}\,dx$$
converges with
$$\int_1^\infty \frac{1}{1+x^2}\,dx \leq 1. \tag{4.6.19}$$
Moreover, $1+x^2 \geq 1$ for all x, so
$$\frac{1}{1+x^2} \leq 1$$
for all x. Hence
$$\int_0^1 \frac{1}{1+x^2}\,dx \leq \int_0^1 dx = 1. \tag{4.6.20}$$
Putting (4.6.19) and (4.6.20) together, we have
$$\int_0^\infty \frac{1}{1+x^2}\,dx = \int_0^1 \frac{1}{1+x^2}\,dx + \int_1^\infty \frac{1}{1+x^2}\,dx \leq 1+1=2.$$

In the problems for Section 6.5, you will be asked to show that

$$\int_0^\infty \frac{1}{1+x^2}\,dx = \frac{\pi}{2}.$$

Example Since $\sqrt{x}-1 < \sqrt{x}$ for all $x \geq 0$, we see that

$$0 < \frac{1}{\sqrt{x}} < \frac{1}{\sqrt{x}-1}$$

for all $x \geq 1$. Thus, by the previous proposition,

$$\int_2^\infty \frac{1}{\sqrt{x}-1}\,dx$$

diverges since we saw above that

$$\int_2^\infty \frac{1}{\sqrt{x}}\,dx$$

diverges.

Although we will not go into the details, the previous proposition may be generalized as follows.

Proposition Suppose $h(x) \leq f(x) \leq g(x)$ for all x in an interval $[a,\infty)$ and f, g, and h are integrable on $[a,b]$ for all $a < b < \infty$. If both $\int_a^\infty h(x)dx$ and $\int_a^\infty g(x)dx$ converge, then $\int_a^\infty f(x)dx$ converges as well. Moreover, in that case,

$$\int_a^\infty h(x)dx \leq \int_a^\infty f(x)dx \leq \int_a^\infty g(x)dx. \tag{4.6.21}$$

Note that our previous proposition is a special case of this proposition with $h(x)=0$ for all $x \geq a$. As before, similar results hold for integrals on intervals of the form $(-\infty,b]$ and $(-\infty,\infty)$.

Example Since $-1 \leq \sin(x) \leq 1$ for all x, it follows that

$$-\frac{1}{x^2} \leq \frac{\sin(x)}{x^2} \leq \frac{1}{x^2}$$

for all $x \geq 1$. Moreover,

$$\int_1^\infty \frac{1}{x^2}\,dx = 1$$

and

$$\int_1^\infty -\frac{1}{x^2}\,dx = -\int_1^\infty \frac{1}{x^2}\,dx = -1.$$

Hence it follows that

$$\int_1^\infty \frac{\sin(x)}{x^2}\,dx$$

converges and

$$-1 \le \int_1^\infty \frac{\sin(x)}{x^2}\,dx \le 1.$$

After noticing that for any function f, $-|f(x)| \le f(x) \le |f(x)|$ for all values of x, the following proposition is a special case of the previous proposition.

Proposition If f is defined on $[a,\infty)$ and $\int_a^\infty |f(x)|dx$ converges, then $\int_a^\infty f(x)dx$ converges.

Example Another way to see that

$$\int_1^\infty \frac{\sin(x)}{x^2}\,dx$$

converges is to note that

$$\int_1^\infty \left|\frac{\sin(x)}{x^2}\right|\,dx$$

converges since

$$0 \le \left|\frac{\sin(x)}{x^2}\right| \le \frac{|\sin(x)|}{x^2} \le \frac{1}{x^2}$$

for all $x \ge 1$.

Once again, similar results hold for integrals on intervals of the form $(-\infty, b]$ and $(-\infty,\infty)$.

We now consider another extension to our definition of the definite integral. Suppose the function f is defined on the interval $(a, b]$ with

$$\lim_{x\to a^+} |f(x)| = \infty.$$

If f is integrable on every interval of the form $[c, b]$ with $a < c < b$, then we may, analogous to our earlier definitions, define

$$\int_a^b f(x)dx = \lim_{c\to a^+} \int_c^b f(x)dx, \qquad (4.6.22)$$

provided the limit exists. See Figure 4.6.6.

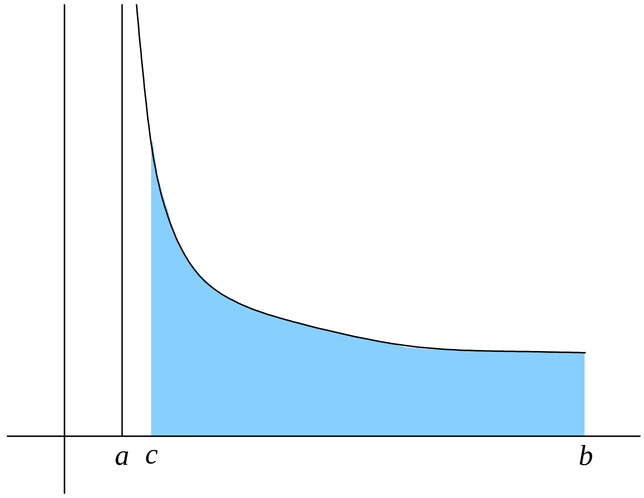

Figure 4.6.6 Area over $(a, b]$ is the area over $[c, b]$ as c approaches a

Definition If f is defined on the interval $(a, b]$ with

$$\lim_{x \to a^+} |f(x)| = \infty,$$

and is integrable on every interval of the form $[c, b]$ with $a < c < b$, then we define

$$\int_a^b f(x)dx = \lim_{c \to a^+} \int_c^b f(x)dx, \tag{4.6.23}$$

provided the limit exists. Similarly, if f is defined on the interval $[a, b)$ with

$$\lim_{x \to b^-} |f(x)| = \infty,$$

and is integrable on every interval of the form $[a, c]$ with $a < c < b$, then we define

$$\int_a^b f(x)dx = \lim_{c \to b^-} \int_a^c f(x)dx, \tag{4.6.24}$$

provided the limit exists. Finally, if f is defined on $[a, d)$ and $(d, b]$ with either

$$\lim_{x \to d^-} |f(x)| = \infty,$$

or

$$\lim_{x \to d^+} |f(x)| = \infty,$$

or both, and f is integrable on all intervals of the form $[a, c]$ with $a < c < d$ and of the form $[c, b]$ with $d < c < b$, then we define

$$\int_a^b f(x)dx = \int_a^d f(x)dx + \int_d^b f(x)dx, \tag{4.6.25}$$

provided both the integrals on the right exist. In each case where the appropriate limit exists, we say the integral converges; otherwise, the integral is said to diverge.

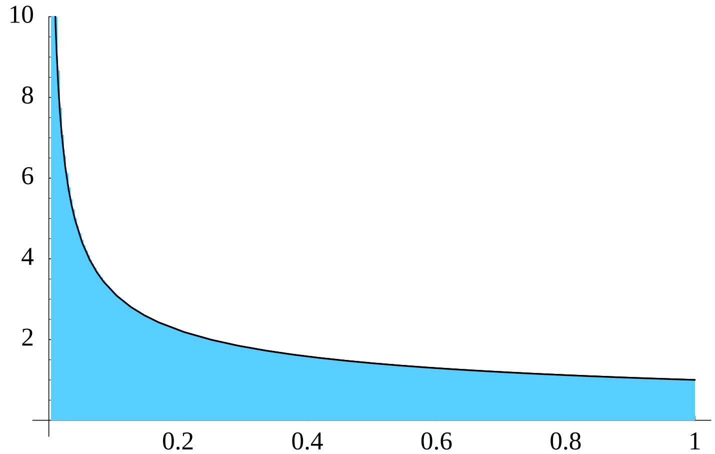

Figure 4.6.7 Region beneath $y = \dfrac{1}{\sqrt{x}}$ over $[0, 1]$

Example The integral

$$\int_0^1 \frac{1}{\sqrt{x}}\,dx$$

converges since

$$\int_0^1 \frac{1}{\sqrt{x}} = \lim_{c\to 0^+}\int_c^1 \frac{1}{\sqrt{x}}\,dx = \lim_{c\to 0^+} 2\sqrt{x}\Big|_c^1 = \lim_{c\to 0^+}(2 - 2\sqrt{c}) = 2.$$

See Figure 4.6.7.

Example The integral

$$\int_0^1 \frac{1}{x^2}\,dx$$

diverges since

$$\int_0^1 \frac{1}{x^2}\,dx = \lim_{c\to 0^+}\int_0^1 \frac{1}{x^2}\,dx = \lim_{c\to 0^+} -\frac{1}{x}\Big|_c^1 = \lim_{c\to 0^+}\left(-1 + \frac{1}{c}\right) = \infty.$$

See Figure 4.6.8.

Example The integral

$$\int_0^2 \frac{1}{(x-1)^{\frac{2}{3}}}\,dx$$

is improper since

$$\lim_{x\to 1^-} \frac{1}{(x-1)^{\frac{2}{3}}} = \infty$$

and

$$\lim_{x\to 1^+} \frac{1}{(x-1)^{\frac{2}{3}}} = \infty.$$

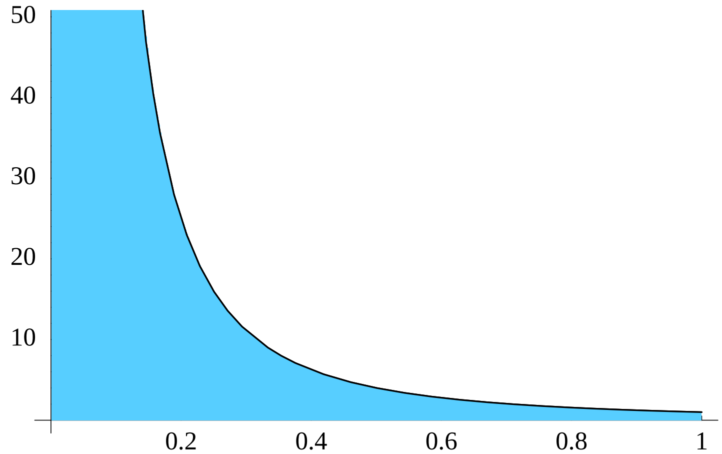

Figure 4.6.8 Region beneath $y = \dfrac{1}{x^2}$ over $[0, 1]$

Moreover, the integral converges since

$$\begin{aligned}\int_0^1 \frac{1}{(x-1)^{\frac{2}{3}}}\,dx &= \lim_{c\to1^-}\int_0^c \frac{1}{(x-1)^{\frac{2}{3}}}\,dx \\ &= \lim_{c\to1^-} 3(x-1)^{\frac{1}{3}}\Big|_0^c \\ &= \lim_{c\to1^-}\left(3(c-1)^{\frac{1}{3}}+3\right) \\ &= 3\end{aligned}$$

and

$$\begin{aligned}\int_1^2 \frac{1}{(x-1)^{\frac{2}{3}}}\,dx &= \lim_{c\to1^+}\int_c^2 \frac{1}{(x-1)^{\frac{2}{3}}}\,dx \\ &= \lim_{c\to1^+} 3(x-1)^{\frac{1}{3}}\Big|_c^2 \\ &= \lim_{c\to1^+}\left(3-3(c-1)^{\frac{1}{3}}\right) \\ &= 3,\end{aligned}$$

which together imply that

$$\int_0^2 \frac{1}{(x-1)^{\frac{2}{3}}}\,dx = \int_0^1 \frac{1}{(x-1)^{\frac{2}{3}}}\,dx + \int_1^2 \frac{1}{(x-1)^{\frac{2}{3}}}\,dx.$$

See Figure 4.6.9.

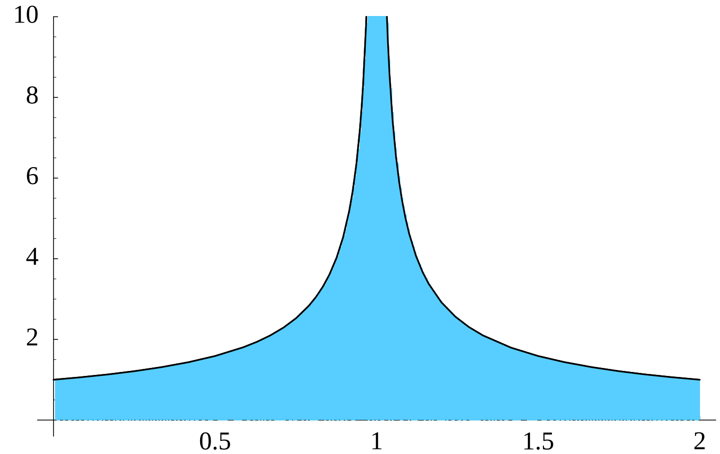

Figure 4.6.9 Region beneath $y = \dfrac{1}{(x-1)^{\frac{2}{3}}}$ over $[0, 2]$

Problems

1. Evaluate the following integrals.

(a) $\displaystyle\int_1^\infty \frac{1}{x^3}\,dx$

(b) $\displaystyle\int_4^\infty \frac{3}{x^7}\,dx$

(c) $\displaystyle\int_{10}^\infty \frac{1}{5x^4}\,dx$

(d) $\displaystyle\int_0^\infty \frac{1}{\sqrt{x+1}}\,dx$

(e) $\displaystyle\int_0^\infty \frac{3}{(2x+3)^2}\,dx$

(f) $\displaystyle\int_0^\infty \sin(x)dx$

2. Evaluate the following integrals.

(a) $\displaystyle\int_{-\infty}^{-2} \frac{3}{x^2}\,dx$

(b) $\displaystyle\int_{-\infty}^\infty \frac{x}{(x^2+4)^4}\,dx$

(c) $\displaystyle\int_{-\infty}^0 \frac{3}{\sqrt{1-x}}\,dx$

(d) $\displaystyle\int_{-\infty}^\infty \frac{5t}{\sqrt{t^2+1}}\,dt$

3. For each of the following, decide, without evaluating, whether the integral converges or diverges.

(a) $\displaystyle\int_1^\infty \frac{1}{x^3+2}\,dx$

(b) $\displaystyle\int_{-\infty}^\infty \frac{1}{x^2+5}\,dx$

(c) $\displaystyle\int_2^\infty \frac{1}{(z^2-2)^{1/3}}\,dz$

(d) $\displaystyle\int_0^\infty \frac{1}{\sqrt{t^4+1}}\,dt$

(e) $\displaystyle\int_1^\infty \frac{\sin^3(t)}{t^2}\,dt$

(f) $\displaystyle\int_\pi^\infty \frac{\cos(z)}{z^5}\,dz$

4. Evaluate the following integrals.

   (a) $\displaystyle\int_0^8 \frac{1}{x^{\frac{1}{3}}}\,dx$

   (b) $\displaystyle\int_0^1 \frac{3}{x^4}\,dx$

   (c) $\displaystyle\int_0^1 \frac{1}{\sqrt{1-x}}\,dx$

   (d) $\displaystyle\int_0^5 \frac{5}{(t-2)^{\frac{2}{5}}}\,dt$

   (e) $\displaystyle\int_{-2}^0 \frac{6}{(z+2)^2}\,dz$

   (f) $\displaystyle\int_{-1}^2 \frac{3}{x^{\frac{1}{3}}}\,dx$

5. (a) Show that
$$\int_1^\infty \frac{1}{x^p}\,dx$$
converges for $p > 1$. Find its value.

   (b) Show that
$$\int_1^\infty \frac{1}{x^p}\,dx$$
diverges for $p < 1$.

6. (a) Show that
$$\int_0^1 \frac{1}{x^p}\,dx$$
converges for $p < 1$. Find its value.

   (b) Show that
$$\int_0^1 \frac{1}{x^p}\,dx$$
diverges for $p > 1$.

7. Let
$$s_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$$
for $n = 1, 2, 3, \ldots$. That is, s_n is the nth partial sum of the harmonic series (see Section 1.3).

   (a) Show that
$$s_n \leq 1 + \int_1^n \frac{1}{x}\,dx$$
for $n = 1, 2, 3, \ldots$. (Hint: Use the right-hand rule to approximate the integral.)

   (b) Show that
$$\int_1^\infty \frac{1}{x}\,dx$$
diverges.

(c) Use a geometric argument to conclude that

$$\int_0^1 \frac{1}{x}\,dx$$

also diverges.

8. For constants $\sigma > 0$ and $\alpha > 0$, the function

$$p(x) = \frac{\alpha\sigma^\alpha}{x^{\alpha+1}},$$

where $x \geq \sigma$, is called a *Pareto distribution.* It is often used in modeling the distribution of incomes or wealth in a population. In the income interpretation, the function

$$P(x) = \int_x^\infty p(t)dt,$$

$x \geq \sigma$, gives the proportion of the population whose income exceeds x. Here σ represents the minimum income of any person in the population and α controls how rapidly the income distribution diminishes as x increases.

(a) Find $P(x)$.

(b) If $\alpha > 1$, the average income of a population described by this model is

$$A = \int_\sigma^\infty xp(x)dx.$$

Find A.

(c) Why is the condition $\alpha > 1$ needed in (b)?

(d) Suppose $\sigma = 10,000$ and $\alpha = 1.2$. Find A, $P(A)$, and $P(2A)$. Interpret the meaning of these values.

(e) Find the general expression for $P(A)$ as a function of α and graph it. Use this graph to interpret the fairness of the income distribution for different values of α.

9. If f is integrable on $[-b, b]$ for all $b > 0$ and

$$\lim_{b\to\infty} \int_{-b}^b f(x)dx$$

exists, then we call

$$I(f) = \lim_{b\to\infty} \int_{-b}^b f(x)dx$$

the *Cauchy integral* of f.

(a) Show that if $\int_{-\infty}^\infty f(x)dx$ converges, then

$$I(f) = \int_{-\infty}^\infty f(x)dx.$$

(b) Find $I(f)$ and $I(g)$ for $f(x) = x$ and $g(x) = \sin(x)$.

(c) Show that the Cauchy integral of f may exist even though $\int_{-\infty}^\infty f(x)dx$ diverges.