Section 5.3

Infinite Series Revisited

Recall from Section 1.3 that for a given sequence $\{a_n\}$, the sequence $\{s_n\}$ with nth term

$$s_n = a_1 + a_2 + a_3 + \cdots + a_n \tag{5.3.1}$$

is called an *infinite series*. An individual term s_n is called a *partial sum* and we say the series is *convergent*, or has a *sum*, if $\lim\limits_{n\to\infty} s_n$ exists. If the series is not convergent, we say it is *divergent*. We write

$$\lim_{n\to\infty} s_n = a_1 + a_2 + a_3 + \cdots + a_n + \cdots = \sum_{n=1}^{\infty} a_n. \tag{5.3.2}$$

Example In Section 1.3 we saw that if $a_n = r^n$, $n = 0, 1, 2, \ldots$, then the associated infinite series, called a geometric series, is convergent if and only if $-1 < r < 1$, in which case

$$\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}. \tag{5.3.3}$$

For example,

$$\sum_{n=0}^{\infty} \frac{1}{3^n} = \frac{1}{1-\frac{1}{3}} = \frac{3}{2}.$$

Geometric series comprise one of the few classes of series for which we can evaluate sums exactly. For most series we can only approximate the sum by computing the partial sums s_n for sufficiently large values of n. However, before this procedure becomes meaningful, we must first know that the series converges. Hence, in this section, as well as in Sections 5.4, 5.5, and 5.6, one of our primary goals will be the development of methods for determining whether a given series converges or diverges.

We begin by considering several basic properties of infinite series. First, suppose we know that both $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are convergent series with

$$\sum_{n=1}^{\infty} a_n = L$$

and

$$\sum_{n=1}^{\infty} b_n = M.$$

If s_n is the nth partial sum of $\sum_{n=1}^{\infty} a_n$, t_n is the nth partial sum of $\sum_{n=1}^{\infty} b_n$, and u_n is the nth partial sum of $\sum_{n=1}^{\infty}(a_n+b_n)$, then $u_n = s_n + t_n$. Thus

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty}(s_n+t_n) = \lim_{n\to\infty} s_n + \lim_{n\to\infty} t_n = L + M. \tag{5.3.4}$$

That is,

$$\sum_{n=1}^{\infty}(a_n+b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n. \tag{5.3.5}$$

Proposition If $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ both converge, then $\sum_{n=1}^{\infty}(a_n+b_n)$ converges and

$$\sum_{n=1}^{\infty}(a_n+b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n. \tag{5.3.6}$$

Similarly, $\sum_{n=1}^{\infty}(a_n-b_n)$ converges and

$$\sum_{n=1}^{\infty}(a_n-b_n) = \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{\infty} b_n. \tag{5.3.7}$$

Example From our results above, it follows that

$$\sum_{n=0}^{\infty}\left(\frac{1}{3^n}+\frac{1}{5^n}\right) = \sum_{n=0}^{\infty}\frac{1}{3^n} + \sum_{n=0}^{\infty}\frac{1}{5^n} = \frac{1}{1-\frac{1}{3}} + \frac{1}{1-\frac{1}{5}} = \frac{3}{2}+\frac{5}{4} = \frac{11}{4}.$$

Now suppose $\sum_{n=1}^{\infty} a_n$ is a convergent series,

$$\sum_{n=1}^{\infty} a_n = L,$$

and k is any constant. If s_n is the nth partial sum of $\sum_{n=1}^{\infty} a_n$ and t_n is the nth partial sum of the series $\sum_{n=1}^{\infty} ka_n$, then $t_n = ks_n$. Hence

$$\lim_{n\to\infty} t_n = \lim_{n\to\infty} ks_n = k\lim_{n\to\infty} s_n = kL. \tag{5.3.8}$$

That is,

$$\sum_{n=1}^{\infty} ka_n = k\sum_{n=1}^{\infty} a_n. \tag{5.3.9}$$

Proposition If $\sum_{n=1}^{\infty} a_n$ converges and k is any constant, then $\sum_{n=1}^{\infty} a_n$ converges and

$$\sum_{n=1}^{\infty} ka_n = k\sum_{n=1}^{\infty} a_n. \tag{5.3.10}$$

Example We have

$$\sum_{n=1}^{\infty} \frac{10}{2^n} = \frac{10}{2} \sum_{n=1}^{\infty} \frac{1}{2^{n-1}} = 5 \sum_{n=0}^{\infty} \frac{1}{2^n} = 5 \left(\frac{1}{1-\frac{1}{2}} \right) = 10.$$

Notice that if $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} ka_n$ must also diverge for any constant $k \neq 0$. This follows because, if, on the contrary, $\sum_{n=1}^{\infty} ka_n$ converged, then, by the previous proposition, so would $\sum_{n=1}^{\infty} a_n$ since

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{1}{k}(ka_n). \tag{5.3.11}$$

Proposition If $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} ka_n$ diverges for any $k \neq 0$.

Example In Section 1.3 we saw that the harmonic series

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

diverges. It follows that both

$$\sum_{n=1}^{\infty} \frac{1}{3n} = \sum_{n=1}^{\infty} \frac{1}{3} \left(\frac{1}{n} \right)$$

and

$$\sum_{n=1}^{\infty} \frac{9}{20n} = \sum_{n=1}^{\infty} \frac{9}{20} \left(\frac{1}{n} \right)$$

are divergent series.

It is also important to note that since

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{m-1} a_n + \sum_{n=m}^{\infty} a_n \tag{5.3.12}$$

for any positive integer m, the series $\sum_{n=1}^{\infty} a_n$ converges if and only if the series $\sum_{n=m}^{\infty} a_n$ converges. In other words, convergence or divergence of a series is never determined by the behavior of any finite number of terms.

Example It follows from the previous example that

$$\sum_{n=200}^{\infty} \frac{9}{20n}$$

diverges.

Example The series

$$\sum_{n=4}^{\infty} \frac{3}{5^n}$$

converges. Moreover,

$$\sum_{n=4}^{\infty} \frac{3}{5^n} = \sum_{n=4}^{\infty} \frac{3}{5^4}\left(\frac{1}{5^{n-4}}\right) = \frac{3}{625}\sum_{n=0}^{\infty} \frac{1}{5^n} = \frac{3}{625}\left(\frac{1}{1-\frac{1}{5}}\right) = \frac{3}{500}.$$

Now suppose the series $\sum_{n=1}^{\infty} a_n$ converges with

$$\sum_{n=1}^{\infty} a_n = L.$$

Let $s_n = a_1 + a_2 + a_3 + \cdots + a_n$ be the nth partial sum of $\sum_{n=1}^{\infty} a_n$. Now

$$a_n = s_n - s_{n-1},$$

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} \sum_{i=1}^{n} a_i = \sum_{i=1}^{\infty} a_i = L,$$

and

$$\lim_{n\to\infty} s_{n-1} = \lim_{n\to\infty} \sum_{i=1}^{n-1} a_i = \sum_{i=1}^{\infty} a_i = L,$$

Hence

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} (s_n - s_{n-1}) = \lim_{n\to\infty} s_n - \lim_{n\to\infty} s_{n-1} = L - L = 0. \tag{5.3.13}$$

That is, the nth term of a convergent series must have a limit of 0.

Proposition If $\sum_{n=1}^{\infty} a_n$ converges, then

$$\lim_{n\to\infty} a_n = 0. \tag{5.3.14}$$

Note that this result only demonstrates a consequence of a series converging, and so does not provide a criterion to determine convergence. However, it may be useful in showing that certain series are divergent. Namely, if either the sequence $\{a_n\}$ does not have a limit or

$$\lim_{n\to\infty} a_n \neq 0,$$

then the series $\sum_{n=1}^{\infty} a_n$ must diverge. This result is often called the *nth term test for divergence.*

Example The series

$$\sum_{n=1}^{\infty} \cos\left(\frac{1}{n}\right)$$

diverges since

$$\lim_{n\to\infty} \cos\left(\frac{1}{n}\right) = \cos(0) = 1.$$

Example The series $\sum_{n=1}^{\infty}(-1)^n$ diverges since $\{(-1)^n\}$ does not have a limit.

Example Note that

$$\lim_{n\to} \frac{1}{n} = 0,$$

yet the harmonic series

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

diverges.

p-series

In the next section we will consider a method for determining the convergence or divergence of a series by comparing a given series with a series which is already known to converge or diverge. In order to make significant use of such a result it is necessary to have a supply of series whose convergence or divergence is already known. So far geometric series are the only series we have studied in any detail. Now we will consider the class of series of the form

$$\sum_{n=1}^{\infty} \frac{1}{n^p}, \tag{5.3.15}$$

where p is a fixed constant. Such series are called *p-series.* The following proposition contains our main result.

Proposition The p-series

$$\sum_{n=1}^{\infty} \frac{1}{n^p}, \tag{5.3.16}$$

converges for $p > 1$ and diverges for $p \leq 1$.

To demonstrate this result, we shall consider four cases. First, suppose $p \leq 0$. Then

$$\lim_{n\to\infty} \frac{1}{n^p} = \begin{cases} \infty, & \text{if } p < 0, \\ 1, & \text{if } p = 0. \end{cases} \tag{5.3.17}$$

Thus the series diverges by the nth term test for divergence.

Next, consider $0 < p < 1$. Note that for any $n > 0$, the partial sum

$$s_n = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \cdots + \frac{1}{n^p}, \tag{5.3.18}$$

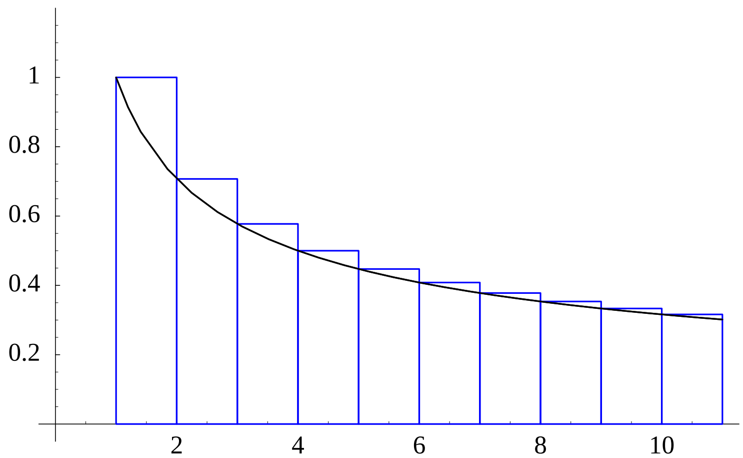

Figure 5.3.1 Rectangles for left-hand rule approximation for $\int_1^{11} \frac{1}{\sqrt{x}}\, dx$

is a left-hand rule approximation, using intervals of length 1, for the integral

$$\int_1^{n+1} \frac{1}{x^p}\, dx. \tag{5.3.19}$$

See Figure 5.3.1 for the case $p = \frac{1}{2}$ and $n = 10$. Since

$$f(x) = \frac{1}{x^p}$$

is a decreasing function on the interval $[1, n+1]$, s_n is an upper sum for the integral (5.3.19), and hence

$$s_n \geq \int_1^{n+1} \frac{1}{x^p}\, dx. \tag{5.3.20}$$

Now

$$\int_1^{n+1} \frac{1}{x^p}\, dx = \left.\frac{x^{1-p}}{1-p}\right|_1^{n+1} = \frac{(n+1)^{1-p} - 1}{1-p}. \tag{5.3.21}$$

Thus

$$s_n \geq \frac{(n+1)^{1-p} - 1}{1-p}. \tag{5.3.22}$$

But, since $1 - p > 0$,

$$\lim_{n\to\infty} \frac{(n+1)^{1-p} - 1}{1-p} = \infty. \tag{5.3.23}$$

Hence $\{s_n\}$ is an unbounded, increasing sequence, and so, from a result in Section 1.2,

$$\lim_{n\to\infty} s_n = \infty. \tag{5.3.24}$$

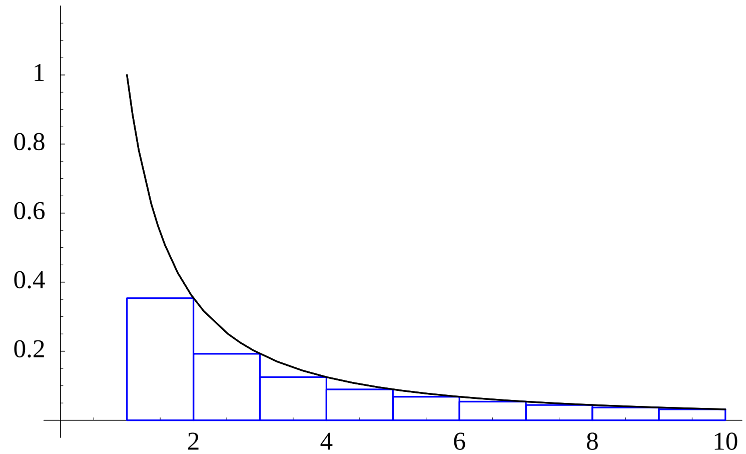

Figure 5.3.2 Rectangles for right-hand rule approximation for $\int_1^{10} \frac{1}{x^{1.5}}\,dx$

In other words,

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

is a divergent series.

For $p = 1$, the p-series is the harmonic series and so diverges.

Finally, consider $p > 1$. If for any integer $n > 1$ we let

$$t_n = \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \cdots + \frac{1}{n^p}, \tag{5.3.25}$$

then t_n is a right-hand rule approximation, using intervals of length 1, for the integral

$$\int_1^n \frac{1}{x^p}\,dx. \tag{5.3.26}$$

See Figure 5.3.2 for the case $p = 1.5$ and $n = 10$. Since

$$f(x) = \frac{1}{x^p}$$

is a decreasing function on the interval $[1, n]$, t_n is a lower sum for the integral (5.3.26), and hence

$$t_n \leq \int_1^n \frac{1}{x^p}\,dx. \tag{5.3.27}$$

Now

$$\begin{aligned} \int_1^n \frac{1}{x^p}\,dx &\le \int_1^\infty \frac{1}{x^p}\,dx \\ &= \lim_{n\to\infty} \int_1^n \frac{1}{x^p}\,dx \\ &= \lim_{n\to\infty} \left.\frac{x^{1-p}}{1-p}\right|_1^n \\ &= \lim_{n\to\infty} \frac{n^{1-p}-1}{1-p} \\ &= \frac{1}{p-1}, \end{aligned} \tag{5.3.28}$$

where the final equality follows from the fact that, since $p > 1$,

$$\lim_{n\to\infty} n^{1-p} = \lim_{n\to\infty} \frac{1}{n^{p-1}} = 0. \tag{5.3.29}$$

Thus

$$t_n \le \frac{1}{p-1}. \tag{5.3.30}$$

Now if s_n is the nth partial sum of

$$\sum_{n=1}^{\infty} \frac{1}{n^p},$$

then $s_1 = 1$ and $s_n = 1 + t_n$, $n = 2, 3, 4, \ldots$. Hence

$$s_n \le 1 + \frac{1}{p-1} = \frac{p}{p-1} \tag{5.3.31}$$

for $n = 1, 2, 3, \ldots$. Thus $\{s_n\}$ is a bounded, increasing sequence, and so, from a result in Section 1.2, must have a limit. That is, $\lim\limits_{n\to\infty} s_n$ exists and

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

is a convergent series.

Example The series

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$$

diverges because it is a p-series with $p = \frac{1}{2}$. Moreover, it now follows from our earlier results that the series

$$\sum_{n=1}^{\infty} \frac{3}{2\sqrt{n}}$$

and

$$\sum_{n=10}^{\infty} \frac{1}{\sqrt{n}}$$

both diverge as well.

Example The series

$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

converges because it is a p-series with $p = 2$. Similar to our last example, it now follows from our earlier results that the series

$$\sum_{n=1}^{\infty} \frac{35}{6n^2}$$

and

$$\sum_{n=20}^{\infty} \frac{7}{5n^2}$$

both converge as well. Moreover, from (5.3.31), we know that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \leq \frac{2}{2-1} = 2.$$

In Problem 5 in Section 4.6, you were asked to show that the integral

$$\int_1^{\infty} \frac{1}{x^p}\, dx$$

diverges for $p < 1$ and converges for $p > 1$. In Section 6.2 we will see that

$$\int_1^{\infty} \frac{1}{x}\, dx$$

diverges as well (see also Problem 5 of this section). Combining these facts with our results about p-series, it follows that

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

converges if and only if

$$\int_1^{\infty} \frac{1}{x^p}\, dx$$

converges. This should not be surprising considering the intimate relationship we have seen between the partial sums of the series and the left-hand and right-hand rule approximations for the integral. The essential ingredient in making these connections was that the function

$$f(x) = \frac{1}{x^p}$$

is continuous, positive, and decreasing on the interval $[1, \infty)$ when $p > 0$. In fact, it can be shown, using arguments similar to those given above, that if g is a continuous, decreasing function on $[1, \infty)$ with $g(x) > 0$ for all $x \geq 1$, then

$$\sum_{n=1}^{\infty} g(n)$$

converges if and only if

$$\int_{1}^{\infty} g(x)dx$$

converges. You are asked to verify this result, known as the *integral test*, in Problem 4.

Problems

1. For each of the following infinite series, decide whether the series converges or diverges and explain your answer. If the series is a convergent geometric series, find its sum.

   (a) $\displaystyle\sum_{n=0}^{\infty} \frac{3}{5^n}$

   (b) $\displaystyle\sum_{n=0}^{\infty} \left(4 - \frac{1}{2^n}\right)$

   (c) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{5}{2^n} + \frac{2}{3^n}\right)$

   (d) $\displaystyle\sum_{n=1}^{\infty} (-3)^n$

   (e) $\displaystyle\sum_{n=1}^{\infty} \left(1 - \frac{1}{n}\right)$

   (f) $\displaystyle\sum_{n=1}^{\infty} n \sin\left(\frac{1}{n}\right)$

   (g) $\displaystyle\sum_{n=10}^{\infty} \frac{(-1)^n}{1000}$

   (h) $\displaystyle\sum_{n=2}^{\infty} \left(-\frac{3}{7}\right)^n$

2. For each of the following infinite series, decide whether the series converges or diverges and explain your answer.

   (a) $\displaystyle\sum_{n=1}^{\infty} \frac{4}{n^3}$

   (b) $\displaystyle\sum_{n=1}^{\infty} \frac{2}{n^{15}}$

   (c) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{n^2} - \frac{5}{n^4}\right)$

   (d) $\displaystyle\sum_{n=21}^{\infty} \frac{3}{\sqrt{n}}$

   (e) $\displaystyle\sum_{n=1}^{\infty} n^{-\frac{1}{3}}$

   (f) $\displaystyle\sum_{n=1}^{\infty} \sqrt{\frac{1}{n^3}}$

   (g) $\displaystyle\sum_{n=3}^{\infty} \frac{1}{234\sqrt{n}}$

   (h) $\displaystyle\sum_{n=5}^{\infty} \frac{210 - \sqrt{n}}{n^2}$

3. Give an example of divergent series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ for which the series
$$\sum_{n=1}^{\infty}(a_n + b_n)$$
converges.

4. Prove the integral test. That is, show that if g is a continuous decreasing function on $[1, \infty)$ with $g(x) > 0$ for all $x \geq 1$, then
$$\sum_{n=1}^{\infty} g(n)$$
converges if and only if
$$\int_1^{\infty} g(x)dx$$
converges.

5. Use the integral test to determine the convergence or divergence of each of the following.

   (a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+3}$ (b) $\displaystyle\sum_{n=2}^{\infty} \frac{5}{\sqrt{n-1}}$

   (c) $\displaystyle\int_1^{\infty} \frac{1}{x}\,dx$ (d) $\displaystyle\sum_{n=1}^{\infty} \frac{3n}{\sqrt{n^2-1}}$

6. Find three different examples of divergent series $\sum_{n=1}^{\infty} a_n$ with the property that $\lim_{n \to \infty} a_n = 0$.

7. The following argument has been used to show that
$$\sum_{n=0}^{\infty}(-1)^n = \frac{1}{2}:$$
Let
$$L = \sum_{n=0}^{\infty}(-1)^n.$$
Then
$$L = \sum_{n=0}^{\infty}(-1)^n = 1 + \sum_{n=1}^{\infty}(-1)^n = 1 - \sum_{n=0}^{\infty}(-1)^n = 1 - L.$$
Thus $L = 1 - L$, and so $L = \frac{1}{2}$. Where is the fallacy in this argument?