1 Introduction; Integration by Parts

September 11-12

Traditionally Calculus I covers "Differential Calculus" and Calculus II covers "Integral Calculus." You have already seen the Riemann integral and certain applications in your first semester of calculus. Calculus II starts off with a collection of techniques of integration, and then moves on to polar coordinates, parametric equations and infinite sequences and series. While this looks like a rather mixed bag of topics, it is also possible to think of these topics as helping us integrate functions.

It is an unfair asymmetry of calculus that integrating functions is a much more difficult problem than differentiating functions. You can pretty much differentiate any function you can write down simply by applying all the various differentiation rules you learned in Calculus I. The Sum Rule, Product Rule, Chain Rule etc. all allow you to calculate derivatives of fantastically complicated expressions. While we may know that an antiderivative for any continuous function on a closed interval exists, what we are usually after is an explicit antiderivative for the function, which is useful for us in calculating definite integrals via the Fundamental Theorem of Calculus.

Consider the following functions:

- $\sin x$
- $x \sin x^2$
- $x \sin x$
- $\frac{\sin x}{x}$
- $\sin x^2$

We have seen a couple of methods already; inversion of the basic derivative formulas give us some integral formulas, so for example $\frac{d}{dx}(-\cos x) = \sin x$ means that $\int \sin x dx = -\cos x + C$. Also, we can use inversions of more general formulas to give us general techniques; the $u-$substitution method is just the Chain Rule expressed as an integral formula (see p. 299 in your text). This allows us to evaluate the second integral in the list above. A similar inversion of the Product Rule will give us a method known as Integration by

Parts. We will use this to evaluate the third integral in the list after deriving the formula.

What about the last two integrals? It turns out that no matter what methods we use, we cannot find an elemantary antiderivative for either of these. In order to deal with things like this, it will be necessary to develop an entirely different way to represent functions — power series. The development of power series representations of functions will take the better part of this semester. For now however we will consider several common techniques of integration.

Let's take a look at the Product Rule for differentiation:

$$(fg)'(x) = f(x)g'(x) + f'(x)g(x).$$

Integrating, we get

$$\int (fg)'(x)dx = \int f(x)g'(x)dx + \int f'(x)g(x)dx$$

or

$$f(x)g(x) + C = \int f(x)g'(x)dx + \int f'(x)g(x)dx$$

Solving for $\int f(x)g'(x)dx$ gives

$$\int f(x)g'(x)dx = f(x)g(x) - \int f'(x)g(x)dx$$

As a notational convenience, we let $u = f(x)$ and $dv = g'(x)dx$ so that $du = f'(x)dx$ and $v = g(x)$. Using this notation, we may write

$$\int udv = uv - \int vdu$$

This is the integration by parts formula.

Example $\int x \sin x dx$

We choose u to be x, and dv to be $\sin x\, dx$. Differentiating $u = x$ gives $\frac{du}{dx} = 1$, or $du = dx$. Integrating $dv = \sin x\, dx$ gives $v = -\cos x$. (We may take the constant of integration to be zero). Notice that the combination of u and dv **must** completely absorb the entire integrand.

$$\begin{aligned} \int x \sin x \, dx &= -x\cos x - \int -\cos x \, dx \\ &= -x\cos x + \sin x + C \end{aligned}$$

As an exercise, try the choice $u = \sin x$, $dv = dx$ and see what you get. Will this way work just as well as the choice above?

Integration by parts gives us some very useful antiderivatives. Some applications of this technique are straightforward, as in the last example last class. Others are a little more devious: for example, here is a simple-looking integral which might not seem a candidate for integration by part rights away, since it doesn't look like a product:

Example: $\int \ln x \, dx$

Let $u = \ln x$ and $dv = dx$, so $du = \frac{1}{x}dx$ and $v = x$.
Now,

$$\begin{aligned} \int \ln x dx &= x \ln x - \int dx \\ &= x\ln x - x + C \end{aligned}$$

Another lovely application:

Example: $\int \tan^{-1} x dx$

Let $u = \tan^{-1} x$ and $dv = dx$, so $du = \frac{dx}{1+x^2}$ and $v = x$. Now,

$$\begin{aligned} \int \tan^{-1} x dx &= x\tan^{-1} x - \int \frac{x dx}{1+x^2} \\ &= x\tan^{-1} x - \frac{1}{2}\ln \mid 1 + x^2 \mid + C \\ &= x\tan^{-1} x - \frac{1}{2}\ln(1+x^2) + C \end{aligned}$$

Sometimes it may be necessary to apply the technique more than once to arrive at a complete answer. For example, it is easy to see with the correct choices of u and dv that the integrals $\int x^2 \sin x \, dx$ and $\int x \ln x \, dx$ will reduce to integrals which we have already done by parts. Other examples are a bit surprising:

Example: $\int e^x \sin x\, dx$

We let $u = e^x$ and $dv = \sin x\, dx$. Thus, $du = e^x dx$ and $v = -\cos x$. A first application of the technique gives us

$$\int e^x \sin x\, dx = -e^x \cos x + \int e^x \cos x\, dx$$

At first blush it seems that our try here was a failure, since we ended up with an integral which is the same difficulty as what we started with and which we also don't know how to do. However, let's attempt this second integral, again by parts, setting $U = e^x$ and $dV = \cos x\, dx$.:

$$\int e^x \cos x\, dx = e^x \sin x - \int e^x \sin x\, dx$$

Substituting in this value for $\int e^x \cos x\, dx$ in the previous equation gives us

$$\int e^x \sin x\, dx = -e^x \cos x + e^x \sin x - \int e^x \sin x\, dx$$

As awful as this looks, we notice that $\int e^x \sin x\, dx$ apepars with different signs on both sides of the equation; we can add this to both sides to get:

$$2\int e^x \sin x\, dx = -e^x \cos x + e^x \sin x + C'$$

or

$$\frac{1}{2}\int e^x \sin x\, dx = \frac{1}{2}e^x(\sin x - \cos x) + C$$

This kind of calculation arises more often than you might think, so please remember this trick and the integrals which you can calculate using it. Here's another:

Example: $\int \sec^3 x\, dx$

Let $u = \sec x$ and $dv = \sec^2 x\, dx$. This seems like a very tricky thing to do until you remember that $\sec^2 x$ is something that you know how to integrate! We get $du = \sec x \tan x\, dx$ and $v = \tan x$.

$$
\begin{aligned}
\int \sec^3 x\,dx &= \sec x \tan x - \int \sec x \tan^2 x\,dx \\
&= \sec x \tan x - \int \sec x(\sec^2 x - 1)\,dx \\
&= \sec x \tan x - \int \sec^3 x\,dx + \int \sec x\,dx \\
&= \sec x \tan x - \int \sec^3 x\,dx + \ln|\sec x + \tan x| + C'
\end{aligned}
$$

Combining the $\int \sec^3 x\,dx$ terms into the left-hand side we get

$$2\int \sec^3 x\,dx = \sec x \tan x + \ln|\sec x + \tan x| + C'$$

or

$$\int \sec^3 x\,dx = \frac{1}{2}\sec x \tan x + \frac{1}{2}\ln|\sec x + \tan x| + C$$

where $C = \frac{1}{2}C'$.

Note: Here we are using the fact that $\int \sec x\,dx = \ln|\sec x + \tan x| + C$. This can be calculated by multiplying the integrand on both top and bottom by $\sec x + \tan x$ and observing that the new top is now the derivative of the bottom.