Lecture 31: Substitution

31.1 Substitution in indefinite integrals

Example $\int 2x\sqrt{1+x^2}dx = \frac{2}{3}(1+x^2)^{\frac{3}{2}} + c$

Note that the previous example is of the form

$$\int f(g(x))g'(x)dx,$$

where $f(x) = \sqrt{x}$ and $g(x) = 1 + x^2$. In general, if F is an antiderivative of f, then, by the chain rule,

$$\int f(g(x))g'(x)dx = F(g(x)) + c.$$

Using Leibniz notation, if $u = g(x)$, then

$$\int f(u)\frac{du}{dx}dx = F(u) + c.$$

But $\int f(u)du = F(u) + c$, so we have

$$\int f(\underbrace{g(x)}_{u})\underbrace{g'(x)dx}_{du} = \int f(u)\frac{du}{dx}dx = \int f(u)du.$$

Note that, symbolically, we may think of substituting u for $g(x)$ and du for $g'(x)dx$.

Example To evaluate $\int 2x\sqrt{1+x^2}dx$ we make the substitution $u = 1+x^2$, from which we have $\frac{du}{dx} = 2x$, or $du = 2xdx$. Hence

$$\int 2x\sqrt{1+x^2}dx = \int \sqrt{u}du = \frac{2}{3}u^{\frac{3}{2}} + c = \frac{2}{3}(1+x^2)^{\frac{3}{2}} + c.$$

Example To evaluate $\int x\sqrt{1+x^2}dx$, we make the substitution

$$u = 1 + x^2$$
$$du = 2xdx \Rightarrow \frac{1}{2}du = xdx.$$

Notice what we have done to account for the fact that the derivative of the substitution differs from x by a constant factor. Then we have

$$\int x\sqrt{1+x^2}dx = \frac{1}{2}\int \sqrt{u}du = \frac{1}{3}u^{\frac{3}{2}} + c = \frac{1}{3}(1+x^2)^{\frac{3}{2}} + c.$$

Example To evaluate $\int x^2 \sin(4x^3)dx$, we make the substitution

$$u = 4x^3$$
$$du = 12x^2 dx \Rightarrow \frac{1}{12}du = x^2 dx.$$

Then

$$\int x^2 \sin(4x^3)dx = \frac{1}{12}\int \sin(u)du = -\frac{1}{12}\cos(u) + c = -\frac{1}{12}\cos(4x^3) + c.$$

Example To evaluate $\int \sin^2(x)\cos(x)dx$, we make the substitution

$$u = \sin(x)$$
$$du = \cos(x)dx.$$

Then

$$\int \sin^2 \cos(x)dx = \int u^2 du = \frac{1}{3}u^3 + c = \frac{1}{3}\sin^3(x) + c.$$

Example To evaluate $\int \sqrt{4x+5}\ dx$, we make the substitution

$$u = 4x+5$$
$$du = 4dxdx \Rightarrow \frac{1}{4}du = dx.$$

Then

$$\int \sqrt{4x+5}\ dx = \frac{1}{4}\int \sqrt{u}du = \frac{2}{12}u^{\frac{3}{2}} + c = \frac{1}{6}(4x+5)^{\frac{3}{2}} + c.$$

Example To evaluate $\int x\sqrt{1+x}\ dx$, we make the substitution

$$u = 1+x$$
$$du = dx.$$

Then, using $x = u - 1$,

$$\begin{aligned}\int x\sqrt{1+x}\ dx &= \int (u-1)\sqrt{u}du \\ &= \int u^{\frac{3}{2}}du - \int \sqrt{u}du \\ &= \frac{2}{5}u^{\frac{5}{2}} - \frac{2}{3}u^{\frac{3}{2}} + c \\ &= \frac{2}{5}(1+x)^{\frac{5}{2}} - \frac{2}{3}(1+x)^{\frac{3}{2}} + c.\end{aligned}$$

31.2 Substitution in a definite integral

For a definite integral, if F is an antiderivative of f, we have

$$\int_a^b f(g(x))g'(x)dx = F(g(x))\Big|_a^b = F(g(b)) - F(g(a)) = \int_{g(a)}^{g(b)} f(u)du.$$

Example To evaluate $\displaystyle\int_0^{\frac{\pi}{4}} \cos^2(2x)\sin(2x)dx$, we make the substitution

$$\begin{aligned}u &= \cos(2x) \\ du &= -2\sin(x)dx \Rightarrow -\frac{1}{2}du = \sin(2x)dx.\end{aligned}$$

Then

$$\int_0^{\frac{\pi}{4}} \cos^2(2x)\sin(2x)dx = -\frac{1}{2}\int_1^0 u^2du = \frac{1}{2}\int_0^1 u^2du = \frac{1}{6}u^3\Big|_0^1 = \frac{1}{6}.$$

Example To evaluate $\displaystyle\int_0^1 x(1+x)^{10}dx$, we make the substitution

$$\begin{aligned}u &= 1+x \\ du &= dx.\end{aligned}$$

Then

$$\begin{aligned}
\int_0^1 x(1+x)^{10}dx &= \int_1^2 (u-1)u^{10}du \\
&= \int_1^2 u^{11}du - \int_1^2 u^{10}du \\
&= \frac{1}{12}u^{12}\Big|_1^2 - \frac{1}{11}u^{11}\Big|_1^2 \\
&= \frac{2^{12}-1}{12} - \frac{2^{11}-1}{11} \\
&= \frac{4095}{12} - \frac{2047}{11} \\
&= \frac{20481}{132} \\
&= \frac{6827}{44}.
\end{aligned}$$