# INTEGRATION TECHNIQUES
## Chapter 9

Procedures for working with integrands: 9.1 p497
1) Expand (carry out operations)
2) Separate numerator (around + or - signs)
3) Complete the square (in denominator)
4) Divide (if power in numerator is equal to or greater than power in denominator)
5) Create new terms in numerator by adding and subtracting a number.
6) Use trigonometric identities.
7) Multiply by conjugate/conjugate.

**Integration by Parts**: 9.2 p.499

$$\int u\,dv = uv - \int v\,du$$

Try letting $dv$ be the most complicated portion of the integrand that fits an integration formula.

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9.3 p509

If the power of the **sine** is **odd**, put in the form:

$$\int (\sin^2 x)^k \cos^n x \sin x\,dx$$

and convert to: $\int (1-\cos^2 x)^k \cos^n x \sin x\,dx$

Reserve the $\sin x\,dx$ term, multiply, and let $u = \cos x$.

If the power of the **cosine** is **odd**, put in the form:

$$\int \sin^m x\,(\cos^2 x)^k \cos x\,dx$$

and convert to: $\int \sin^m x\,(1-\sin^2 x)^k \cos x\,dx$

Reserve the $\cos x\,dx$ term, multiply, and let $u = \sin x$.

If the powers of both the **sine** and **cosine** are **even**, use:

$\sin^2 x = \dfrac{1-\cos 2x}{2}$ and $\cos^2 x = \dfrac{1+\cos 2x}{2}$

to first convert the integrand to odd powers of cosine.

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9.3 p509

If the power of the **secant** is **even**, put in the form:

$$\int (\sec^2 x)^k \tan^n x \sec^2 x\,dx$$

and convert to: $\int (1+\tan^2 x)^k \tan^n x \sec^2 x\,dx$

Reserve the $\sec^2 x\,dx$ term, multiply, and let $u = \tan x$.

If the power of the **tangent** is **odd**, put in the form:

$$\int \sec^m x\,(\tan^2 x)^k \sec x \tan x\,dx$$

and convert to: $\int \sec^m x\,(\sec^2 x - 1)^k \sec x \tan x\,dx$

Reserve the $\tan x\,dx$ term, multiply, and let $u = \sec x$.

If there are no secant factors and the power of the **tangent** is **even**, put in the form:

$$\int \tan^n x\,(\tan^2 x)dx$$

and convert to: $\int \tan^n x\,(\sec^2 x - 1)dx$

and then: $\int \tan^n x\,(\sec^2 x)dx - \int \tan^n x\,dx$

If the integral is of the form $\int \sec^m x\,dx$. where $m$ is odd, use integration by parts.

If none of the above applies, try converting to sines and cosines.

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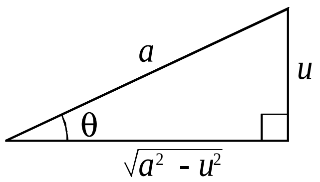
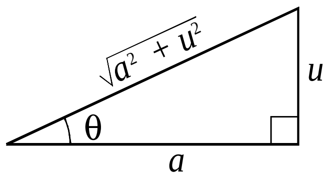
**Trigonometric Substitution**: 9.4 p518 After performing the integration using the substituted values, convert back to terms of $x$ using the right triangles as a guide.

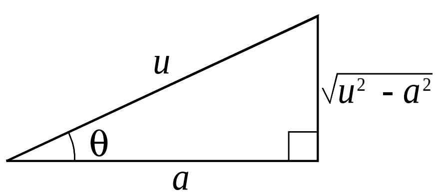
For integrals involving $\sqrt{a^2 - u^2}$, let $u = a\sin\theta$, $du = a\cos\theta\, d\theta$. Then $\sqrt{a^2 - u^2} = a\cos\theta$ where $-\pi/2 \le \theta \le \pi/2$.




For integrals involving $\sqrt{a^2 + u^2}$, let $u = a\tan\theta$, $du = a\sec^2\theta\, d\theta$. Then $\sqrt{a^2 + u^2} = a\sec\theta$ where $-\pi/2 < \theta < \pi/2$.




For integrals involving $\sqrt{u^2 - a^2}$, let $u = a\sec\theta$, $du = a\sec\theta\tan\theta\, d\theta$. Then $\sqrt{u^2 - a^2} = \pm a\tan\theta$ where $0 \le \theta < \pi/2$ or $\pi/2 < \theta \le \pi$. Use the positive value if $u > a$ and the negative value if $u < -a$.




$$\int \sqrt{a^2 - u^2}\, du = \frac{1}{2}\left( a^2 \arcsin\frac{u}{a} + u\sqrt{a^2 - u^2} \right) + C$$

$$\int \sqrt{u^2 - a^2}\, du = \frac{1}{2}\left( u\sqrt{u^2 - a^2} - a^2 \ln\left| u + \sqrt{u^2 - a^2} \right| \right) + C, \quad u > a$$

$$\int \sqrt{u^2 + a^2}\, du = \frac{1}{2}\left( u\sqrt{u^2 + a^2} + a^2 \ln\left| u + \sqrt{u^2 + a^2} \right| \right) + C$$

Partial Fractions: 9.5 p529

The denominator is reduced to simplist terms. Terms with an internal power of 2 require an $x$-term in the numerator. Terms raised to a power require separate fractions with denominators of each power from 1 to the highest power. To determine the values of the new numerators, multiply each by the value needed to achieve the original common denominator, add them together and set this equal to the original numerator.

$$\int \frac{numerator}{(x+a)(x^2+b)(x+c)^2}\, dx = \int \left( \frac{A}{x+a} + \frac{Bx+C}{x^2+b} + \frac{D}{x+c} + \frac{E}{(x+c)^2} \right) dx$$

$$numerator = A(x^2+b)(x+c)^2 + (Bx+C)(x+a)(x+c)^2 + D(x+a)(x^2+b)(x+c) + E(x+a)(x^2+b)$$

Improper integrals with infinite limits of integration: 9.7p.546

$$\int_a^{\infty} f(x)\, dx = \lim_{b\to\infty} \int_a^b f(x)\, dx$$

$$\int_{-\infty}^{b} f(x)\, dx = \lim_{a\to-\infty} \int_a^b f(x)\, dx$$

$$\int_{-\infty}^{\infty} f(x)\, dx = \int_{-\infty}^{c} f(x)\, dx + \int_c^{\infty} f(x)\, dx$$

Improper integrals with an infinite discontinuity: p.549

If $f$ is continuous on the interval $[a, b)$ and has an infinite discontinuity at $b$, then

$$\int_a^b f(x)\, dx = \lim_{c\to b^-} \int_a^c f(x)\, dx$$

if $f$ is continuous on the interval $(a, b]$ and has an infinite discontinuity at $a$, then

$$\int_a^b f(x)\, dx = \lim_{c\to a^+} \int_c^b f(x)\, dx$$

If $f$ is continuous on the interval $[a, b]$, except for some $c$ in $(a, b)$ at which $f$ has an infinite discontinuity, then

$$\int_a^b f(x)\, dx = \int_a^c f(x)\, dx + \int_c^b f(x)\, dx$$

In the first two cases, if the limit exists, then the improper integral **converges**, otherwise, it **diverges**. In the third case, the improper integral on the left **diverges** if either of the improper integrals on the right diverges.

# Section 4.5

# More Techniques of Integration

In the last section we saw how we could exploit our knowledge of the chain rule to develop a technique for simplifying integrals using suitably chosen substitutions. In this section we shall see how we can develop a second technique, called *integration by parts*, using the product rule. Outside of algebraic manipulation and the use of various functional identities, like the trigonometric identities, substitution and parts are the only basic techniques we have available to us for simplifying the process of evaluating an integral.

**Example** Suppose we wish to find $\int x\cos(x)dx$. Since

$$\int \cos(x)dx = \sin(x) + c,$$

we might make an initial guess of $F(x) = x\sin(x)$ for an antiderivative of $f(x) = x\cos(x)$. But, of course, differentiation of $F$, using the product rule, yields

$$F'(x) = x\cos(x) + \sin(x),$$

which differs from the desired result, $f(x)$, by the term $\sin(x)$. However, since

$$\int \sin(x)dx = -\cos(x) + c,$$

we can obtain an antiderivative of $f(x)$ by adding on the term $\cos(x)$ to $F(x)$. That is,

$$G(x) = x\sin(x) + \cos(x)$$

is an antiderivative of $f(x)$ since the derivative of $\cos(x)$ will cancel the $\sin(x)$ term in $F'(x)$. Explicitly,

$$G'(x) = x\cos(x) + \sin(x) - \sin(x) = x\cos(x).$$

Thus

$$\int x\cos(x)dx = x\sin(x) + \cos(x) + c.$$

In general, suppose $f$ and $g$ are differentiable functions and we want to evaluate

$$\int f(x)g'(x)dx. \tag{4.5.1}$$

For example, in our previous example we would have $f(x) = x$ and $g(x) = \sin(x)$. From the product rule we know that

$$\frac{d}{dx}f(x)g(x) = f(x)g'(x) + g(x)f'(x). \tag{4.5.2}$$

Thus, integrating both sides of (4.5.2), we have

$$\int \frac{d}{dx}f(x)g(x) = \int f(x)g'(x)dx + \int g(x)f'(x)dx, \tag{4.5.3}$$

from which it follows that

$$f(x)g(x) = \int f(x)g'(x)dx + \int g(x)f'(x)dx. \tag{4.5.4}$$

Rearranging (4.5.4) gives us

$$\int f(x)g'(x)dx = f(x)g(x) - \int g(x)f'(x)dx. \tag{4.5.5}$$

Applying (4.5.5) to our example, with $f(x) = x$ and $g(x) = \sin(x)$, we have

$$\int x\cos(x)dx = x\sin(x) - \int \sin(x)dx = x\sin(x) + \cos(x) + c.$$

In effect, using (4.5.5), we have replaced the problem of evaluating $\int x\cos(x)dx$ with the simpler problem of evaluating $\int \sin(x)dx$. In general, the success of this method always depends on the integral

$$\int g(x)f'(x)dx \tag{4.5.6}$$

being easier to evaluate than the integral

$$\int f(x)g'(x)dx. \tag{4.5.7}$$

It is common with this technique to let $u = f(x)$ and $v = g(x)$ along with the notation, as we did with substitution,

$$dv = g'(x)dx \tag{4.5.8}$$

and

$$du = f'(x)dx. \tag{4.5.9}$$

With this notation, (4.5.5) becomes

$$\int u dv = uv - \int v du, \tag{4.5.10}$$

the standard form for what is known as integration by parts.

**Example** To evaluate the integral $\int x\sin(x)$ by parts, we must first make a choice for $u$ and $dv$. Here we might choose

$$u = x \quad dv = \sin(x)dx.$$

It follows then that $du = dx$. However, there are many possible choices for $v$; all that we require is that the derivative of $v$ must be $\sin(x)$. The simplest choice is to take $v = -\cos(x)$. Then we have, applying (4.5.10),

$$\int x\sin(x)dx = -x\cos(x) + \int \cos(x)dx = -x\cos(x) + \sin(x) + c.$$

**Example** To evaluate the integral $\int x^2\cos(2x)dx$, we might choose

$$u = x^2 \quad dv = \cos(2x)dx,$$

from which we obtain

$$du = 2xdx \quad v = \frac{1}{2}\sin(2x).$$

Thus

$$\int x^2\cos(2x)dx = \frac{1}{2}x^2\sin(2x) - \int x\sin(2x)dx.$$

This time we do not immediately know the value of the integral on the right, but we know we can find it using integration by parts. Namely, to evaluate $\int x\sin(2x)dx$, we let

$$\begin{aligned} u &= x & dv &= \sin(2x)dx \\ du &= dx & v &= -\frac{1}{2}\cos(2x). \end{aligned}$$

Then

$$\int x\sin(2x)dx = -\frac{1}{2}x\cos(2x) + \frac{1}{2}\int \cos(2x)dx = -\frac{1}{2}x\cos(2x) + \frac{1}{4}\sin(2x) + c.$$

Hence

$$\int x^2\cos(2x)dx = \frac{1}{2}x^2\sin(2x) + \frac{1}{2}x\cos(2x) - \frac{1}{4}\sin(2x) + c.$$

The key to success with integration by parts is in the choice of the parts, $u$ and $dv$. For example, we saw in an example that the choices

$$\begin{aligned} u &= x & dv &= \sin(x)dx \\ du &= dx & v &= -\cos(x) \end{aligned}$$

work well for evaluating $\int x\sin(x)dx$. Alternatively, we could have chosen

$$\begin{aligned} u &= \sin(x) & dv &= xdx \\ du &= \cos(x)dx & v &= \frac{1}{2}x^2, \end{aligned}$$

which would yield

$$\int x\sin(x)dx = \frac{1}{2}x^2\sin(x) - \frac{1}{2}\int x^2\cos(x)dx.$$

All of this is correct, but useless (at least for our present purpose) since the resulting integral on the right is more complicated than the integral with which we started. If we had started to work the problem this way, we would probably stop at this point and rethink our strategy.

**Example** In using integration by parts to evaluate a definite integral, we must remember to evaluate all the pieces of the resulting antiderivative. For example, to evaluate

$$\int_0^{\frac{\pi}{3}} 4x\cos(3x)dx,$$

we might choose

$$\begin{aligned} u &= 4x & dv &= \cos(3x)dx \\ du &= 4dx & v &= \frac{1}{3}\sin(3x). \end{aligned}$$

Then

$$\begin{aligned} \int_0^{\frac{\pi}{3}} 4x\cos(3x)dx &= \frac{4}{3}x\sin(3x)\Big|_0^{\frac{\pi}{3}} - \frac{4}{3}\int_0^{\frac{\pi}{3}}\sin(3x)dx \\ &= (0-0) + \frac{4}{9}\cos(3x)\Big|_0^{\frac{\pi}{3}} \\ &= -\frac{4}{9} - \frac{4}{9} \\ &= -\frac{8}{9}. \end{aligned}$$

**Example** Although integration by parts is most frequently of use when integrating functions involving transcendental functions, such as the trigonometric functions, there are other times when the technique may be used. For example, to compute

$$\int_0^1 x(1+x)^{10}dx,$$

we could use

$$\begin{aligned} u &= x & dv &= (1+x)^{10}dx \\ du &= dx & v &= \frac{1}{11}(1+x)^{11}. \end{aligned}$$

Then

$$\begin{aligned}\int_0^1 x(1+x)^{10}dx &= \frac{1}{11}x(1+x)^{11}\Big|_0^1 - \frac{1}{11}\int_0^1 (1+x)^{11}dx \\ &= \frac{2048}{11} - \frac{1}{132}(1+x)^{12}\Big|_0^1 \\ &= \frac{2048}{11} - \left(\frac{4096}{132} - \frac{1}{132}\right) \\ &= \frac{6827}{44}.\end{aligned}$$

Notice that we could also evaluate this integral using the substitution $u = 1 + x$.

**Miscellaneous examples**

The techniques of substitution and parts are often useful for putting an integral into a form that can be readily evaluated by the Fundamental Theorem of Integral Calculus. The next several examples illustrate how basic trigonometric identities are also useful for rewriting integrals in more easily evaluated forms.

**Example** To evaluate $\int \sin^2(x)dx$, we may use the identity

$$\sin^2(x) = \frac{1-\cos(2x)}{2} \tag{4.5.11}$$

(see Problem 5, Section 2.2). Then

$$\int \sin^2(x)dx = \frac{1}{2}\int (1-\cos(2x))dx = \frac{1}{2}x - \frac{1}{4}\sin(2x) + c.$$

**Example** Similarly, to evaluate

$$\int_0^{\frac{\pi}{4}} \cos^2(2t)dt,$$

we use the identity

$$\cos^2(x) = \frac{1+\cos(2x)}{2}. \tag{4.5.12}$$

Then

$$\begin{aligned}\int_0^{\frac{\pi}{4}} \cos^2(2t)dt &= \frac{1}{2}\int_0^{\frac{\pi}{4}} (1+\cos(4t))dt \\ &= \frac{1}{2}t\Big|_0^{\frac{\pi}{4}} + \frac{1}{8}\sin(4t)\Big|_0^{\frac{\pi}{4}} \\ &= \frac{\pi}{8}.\end{aligned}$$

**Example** To evaluate $\int \sin^2(x)\cos^2(x)dx$, the identity

$$\sin(x)\cos(x) = \frac{1}{2}\sin(2x) \tag{4.5.13}$$

is useful (see Problem 4, Section 2.2.). From it we obtain

$$\begin{aligned}\int \sin^2(x)\cos^2(x)dx &= \int (\sin(x)\cos(x))^2 dx \\ &= \int \left(\frac{1}{2}\sin(2x)\right)^2 dx \\ &= \frac{1}{4}\int \sin^2(2x)dx \\ &= \frac{1}{8}\int (1-\cos(4x))dx \\ &= \frac{1}{8}x - \frac{1}{32}\sin(4x) + c.\end{aligned}$$

**Example** To evaluate $\int \sin^3(x)dx$ we may use the identity

$$\sin^2(x) = 1 - \cos^2(x)$$

to write

$$\sin^3(x) = \sin^2(x)\sin(x) = (1-\cos^2(x))\sin(x).$$

Then the substitution

$$\begin{aligned} u &= \cos(x) \\ du &= -\sin(x)dx \end{aligned}$$

gives us

$$\begin{aligned}\int \sin^3(x)dx &= \int (1-\cos^2(x))\sin(x)dx \\ &= -\int (1-u^2)du \\ &= -u + \frac{1}{3}u^3 + c \\ &= -\cos(x) + \frac{1}{3}\cos^3(x) + c.\end{aligned}$$

This manipulation is useful in evaluating any integral of the form $\int \sin^n(x)dx$ or, in a similar fashion, $\int \cos^n(x)dx$, provided $n$ is a positive odd integer.

**Example** As a final example, note that the identity

$$\tan^2(x) = \sec^2(x) - 1$$

(see Problem 3, Section 2.2) is useful in evaluating

$$\int_0^{\frac{\pi}{4}} \tan^2(x)dx.$$

Namely,

$$\begin{aligned}\int_0^{\frac{\pi}{4}} \tan^2(x)dx &= \int_0^{\frac{\pi}{4}} (\sec^2(x) - 1)dx \\ &= \tan(x)\Big|_0^{\frac{\pi}{4}} - x\Big|_0^{\frac{\pi}{4}} \\ &= 1 - \frac{\pi}{4}.\end{aligned}$$

This concludes our discussion of techniques of integration. As we noted above, there are basically only two techniques for evaluating indefinite integrals, substitution and parts, and even these rely on an ability to reduce a given integral to a form where an antiderivative is recognizable. Hence the situation is not nearly as straightforward as it was for finding derivatives and best affine approximations. For this reason, in the past tables of indefinite integrals were compiled to aid in the evaluation of integrals; when faced with an integral more involved than the basic ones we have investigated in these last two sections, one could hope to find it, or one related to it through a substitution or an integration by parts, in a table. For the most part, tables of integrals have been replaced by computer programs, such as computer algebra systems, which are capable of finding antiderivatives symbolically. Such programs are then able to evaluate definite integrals exactly using the Fundamental Theorem. Although these programs are immensely useful and are an everyday tool for those working with applications of mathematics, one must use them with care. In particular, whenever possible, you should check your answer for reasonableness. Moreover, there are integrals which the system will not be able to evaluate symbolically, either because the given integral is beyond the capabilities of the system, or because a symbolic answer does not even exist. In such cases, one must, of necessity, fall back on numerical approximation techniques.

## Problems

1. Evaluate the following indefinite integrals.

   (a) $\int 3x\sin(x)dx$

   (b) $\int 2x\cos(5x)dx$

   (c) $\int 4x\sin(3x)dx$

   (d) $\int x^2\cos(3x)dx$

   (e) $\int 2x^2\sin(4x)dx$

   (f) $\int x^3\cos(x)dx$

   (g) $\int 3x^3\sin(2x)dx$

   (h) $\int x\sqrt{1+x}\ dx$

2. Evaluate the following definite integrals.

(a) $\int_0^{\pi} 4x\sin(x)dx$ (b) $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 3x\cos(2x)dx$

(c) $\int_0^{\frac{\pi}{3}} 2t\sin(3t)dt$ (d) $\int_0^{\frac{\pi}{2}} x^2\cos(x)dx$

(e) $\int_0^{\frac{\pi}{4}} 2x^2\sin(2x)dx$ (f) $\int_0^{\frac{\pi}{4}} z^3\cos(4z)dz$

3. Evaluate the following indefinite integrals.

(a) $\int \sin^2(2x)dx$ (b) $\int \cos^2(3t)dt$

(c) $\int 5\sin^2(2t)\cos^2(2t)dt$ (d) $\int \sin^3(3x)dx$

(e) $\int 6\cos^3(2z)dz$ (f) $\int \sin^5(t)dt$

(g) $\int \cos^5(2x)dx$ (h) $\int \tan^2(3\theta)d\theta$

4. Evaluate the following definite integrals.

(a) $\int_0^{\pi} \sin^2(x)dx$ (b) $\int_0^{\frac{\pi}{4}} \cos^2(2t)dt$

(c) $\int_0^{\frac{\pi}{2}} 3\sin^2(z)\cos^2(z)dz$ (d) $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^3(t)dt$

(e) $\int_0^{\pi} \sin^3(3t)dt$ (f) $\int_{-\frac{\pi}{8}}^{\frac{\pi}{8}} \tan^2(2t)dt$

5. Evaluate the following integrals using a computer algebra system.

(a) $\int \cos^6(x)dx$ (b) $\int \sin^2(2t)\cos^4(2t)dt$

(c) $\int \sin^4(2t)\cos^4(3t)dt$ (d) $\int \sec^4(3x)dx$

(e) $\int_{-1}^{1} \sqrt{1-x^2}\ dx$ (f) $\int_0^1 x^2\sqrt{1-x^2}\ dx$

(g) $\int_0^{2\pi} \sin^8(2t)dt$ (h) $\int_0^{\frac{\pi}{4}} \tan^6(t)dt$

6. Evaluate the following integrals with any method at your disposal.

(a) $\int_{-\pi}^{\pi} \sin^4(x)dx$ (b) $\int_1^{\pi} \frac{\sin(x)}{x}\ dx$

(c) $\int_0^5 \sin(3x^2)dx$ (d) $\int_0^1 \sqrt{1+x^2}\ dx$

(e) $\int_1^2 \frac{1}{x}\, dx$

(f) $\int_{-2}^{-1} \frac{1}{t}\, dt$

(g) $\int_0^{\pi} \sqrt{5 - 3\sin^2(t)}\, dt$

(h) $\int_0^{2\pi} \frac{1}{\sqrt{1 + \sin^2(t)}}\, dt$

7. If a pendulum of length $b$ is held, at rest, at an angle $\alpha$ from the perpendicular, $0 < \alpha < \pi$, and then released, its period $T$, the time required for one complete oscillation, is given by

$$T = 4\sqrt{\frac{b}{g}} \int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{1 - k^2 \sin^2(\varphi)}}\, d\varphi,$$

where $g = 980$ cm/sec$^2$ (the acceleration due to gravity) and $k = \sin\left(\frac{\alpha}{2}\right)$.

(a) Find the period of a pendulum of length 50 centimeters which is released initially from an angle of $\alpha = \frac{\pi}{3}$.

(b) Repeat (a) for $\alpha = \frac{\pi}{4}, \frac{\pi}{6}, \frac{\pi}{50}$, and $\frac{\pi}{100}$.

(c) In Section 2.2 we noted that for small values of $\alpha$, if $x(t)$ represents the angle the pendulum makes with the perpendicular at time $t$, then, to a good approximation,

$$x(t) = \alpha \cos\left(\sqrt{\frac{g}{b}} t\right).$$

Thus, in this approximation, $x$ has a period of $2\pi\sqrt{\frac{b}{g}}$. For a pendulum of length 50 centimeters, compare this result with your results in parts (a) and (b).

(d) For a pendulum of length 50 centimeters, graph $T$ as a function of $\alpha$ for $-\frac{\pi}{4} \leq \alpha \leq \frac{\pi}{4}$.