Lecture 23: Limits at Infinity

23.1 Limits at infinity

Definition We say the *limit* of $f(x)$ as x approaches infinity is L, denoted $\lim_{x\to\infty} f(x) = L$, if for every $\epsilon > 0$ there exists a number N such that

$$|f(x) - L| < \epsilon$$

whenever $x > N$. We say the *limit* of $f(x)$ as x approaches negative infinity is L, denoted $\lim_{x\to-\infty} f(x) = L$, if for every $\epsilon > 0$ there exists a number N such that

$$|f(x) - L| < \epsilon$$

whenever $x < N$.

Note that if either $\lim_{x\to\infty} f(x) = L$ or $\lim_{x\to-\infty} f(x) = L$, then the line $y = L$ is a horizontal asymptote for the graph of f.

23.2 Examples

Example A basic example is

$$\lim_{x\to\infty} \frac{1}{x} = 0.$$

This follows from the fact that for any $\epsilon > 0$,

$$\frac{1}{x} < \epsilon$$

provided $x > \frac{1}{\epsilon}$. Similarly,

$$\lim_{x\to-\infty} \frac{1}{x} = 0.$$

In general, for any $r > 0$,

$$\lim_{x\to\infty} \frac{1}{x^r} = 0$$

and

$$\lim_{x\to-\infty} \frac{1}{x^r} = 0.$$

Example $$\lim_{x\to\infty} \frac{x}{x^2+1} = \lim_{x\to\infty} \frac{\frac{1}{x}}{1+\frac{1}{x^2}} = \frac{0}{1} = 0$$

Example $\displaystyle\lim_{x\to-\infty} \frac{x}{x^2+1} = \lim_{x\to-\infty} \frac{\frac{1}{x}}{1+\frac{1}{x^2}} = \frac{0}{1} = 0$

Example $\displaystyle\lim_{x\to\infty} \frac{3x^2+4x-6}{8x^2+6x+1} = \lim_{x\to\infty} \frac{3+\frac{4}{x}-\frac{6}{x^2}}{8+\frac{6}{x}+\frac{1}{x^2}} = \frac{3}{8}$

Example $\displaystyle\lim_{x\to\infty} \frac{4x-6}{8x^3+8} = \lim_{x\to\infty} \frac{\frac{4}{x^2}-\frac{6}{x^3}}{8+\frac{8}{x^3}} = \frac{0}{8} = 0$

Example $\displaystyle\lim_{x\to\infty} \frac{8-4x^2}{x+4} = \lim_{x\to\infty} \frac{\frac{8}{x}-4x}{1+\frac{4}{x}} = -\infty$

Example $\displaystyle\lim_{x\to\infty} \frac{x+1}{\sqrt{2x^2+1}} = \lim_{x\to\infty} \frac{1+\frac{1}{x}}{\sqrt{2+\frac{1}{x^2}}} = \frac{1}{\sqrt{2}}$

Example $\displaystyle\lim_{x\to-\infty} \frac{x+1}{\sqrt{2x^2+1}} = \lim_{x\to-\infty} \frac{1+\frac{1}{x}}{-\sqrt{2+\frac{1}{x^2}}} = -\frac{1}{\sqrt{2}}$

Example $\displaystyle\lim_{x\to\infty} (x^3-x^2) = \lim_{x\to\infty} x^3(1-\frac{1}{x}) = \infty$

Example $\displaystyle\lim_{x\to-\infty} (x^3-x^2) = \lim_{x\to-\infty} x^3(1-\frac{1}{x}) = -\infty$

Example $\displaystyle\lim_{x\to\infty} (x-\sqrt{x+1}) = \lim_{x\to\infty} x(1-\sqrt{\frac{1}{x}+\frac{1}{x^2}}) = \infty$

Example $\displaystyle\lim_{x\to-\infty} (x+\sqrt{x^2+1}) = \lim_{x\to-\infty} \frac{x^2-(x^2+1)}{x-\sqrt{x^2+1}} = \lim_{x\to-\infty} \frac{-1}{x-\sqrt{x^2+1}} = 0$

Example Let $f(x) = \dfrac{3x+2}{x-1}$. Then

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{3+\frac{2}{x}}{1-\frac{1}{x}} = 3$$

and

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{3+\frac{2}{x}}{1-\frac{1}{x}} = 3,$$

so the line $y = 3$ is a horizontal asymptote for the graph of f. Moreover,

$$\lim_{x\to 1^-} f(x) = -\infty$$

and

$$\lim_{x \to 1^+} f(x) = \infty,$$

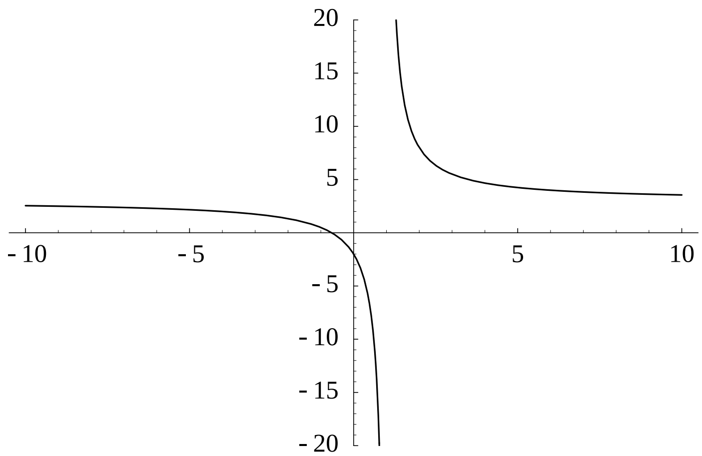
so the line $x = 1$ is a vertical asymptote for the graph of f. Also,

$$f'(x) = \frac{(x-1)(3) - (3x+2)(1)}{(x-1)^2} = -\frac{5}{(x-1)^2}$$

and

$$f''(x) = 10(x-1)^{-3} = \frac{10}{(x-1)^3}.$$

Hence f' and f'' are both undefined at $x = 0$, and f is decreasing on both $(-\infty, 1)$ and on $(1, \infty)$, while the graph of f is concave downward on $(-\infty, 1)$ and concave upward on $(1, \infty)$. With this information, we can sketch the graph of f.



Graph of $f(x) = \dfrac{3x+2}{x-1}$