Lecture 7: Properties of Limits

7.1 Basic properties of limits

Proposition Suppose $\lim_{x \to a} f(x) = L$ and $\lim_{x \to a} g(x) = M$. Then

1. $\lim_{x \to a} (f(x) + g(x)) = L + M$,
2. $\lim_{x \to a} (f(x) - g(x)) = L - M$,
3. $\lim_{x \to a} cf(x) = cL$ for any constant c,
4. $\lim_{x \to a} (f(x)g(x)) = LM$,
5. $\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{L}{M}$ provided $M \neq 0$,
6. For any rational number n, $\lim_{x \to a} (f(x))^n = L^n$.

7.2 Two basic limits

1. For any constant c and number a, $\lim_{x \to a} c = c$.

To show this from the definition, given any $\epsilon > 0$, we need to find $\delta > 0$ such that $|c - c| < \epsilon$ whenever $0 < |x - a| < \delta$. But $|c - c| = 0$ no matter what the value of x is, so we may choose δ to be any positive number.

2. For any number a, $\lim_{x \to a} x = a$.

To show this from the definition, given any $\epsilon > 0$, we need to find $\delta > 0$ such that $|x - a| < \epsilon$ whenever $0 < |x - a| < \delta$. Hence we need only let $\delta = \epsilon$.

7.3 Examples

Example $\lim_{x \to 2} x^2 = (\lim_{x \to 2} x)(\lim_{x \to 2} x) = (2)(2) = 4$

Example More generally, $\lim_{x \to a} x^2 = (\lim_{x \to a} x)(\lim_{x \to a} x) = (a)(a) = a^2$ for any number a.

Example $\lim_{x \to 1} 4x^8 = 4 \lim_{x \to 1} x^8 = (4)(1) = 4$

Example $\lim_{x\to -1}(x^2 - 3x + 4) = \lim_{x\to -1} x^2 - 3\lim_{x\to -1} x + \lim_{x\to -1} 4 = 1 + 3 + 4 = 8$

Theorem If f is a polynomial, then $\lim_{x\to a} = f(a)$ for any number a.

Example $\lim_{x\to 2}(x^3 - 4x + 3) = 8 - 8 + 3 = 3$

Example $\lim_{x\to 1} \frac{x^2+1}{3x+4} = \frac{\lim_{x\to 1}(x^2+1)}{\lim_{x\to 1}(3x+4)} = \frac{2}{7}$

Theorem If f is a rational function defined at a, then $\lim_{x\to a} = f(a)$.

Example $\lim_{x\to 3} \frac{x^2-4}{x+1} = \frac{5}{4}$

Example $\lim_{x\to 2} \frac{x^2-4}{x-2} = \lim_{x\to 2} \frac{(x-2)(x+2)}{x-2} = \lim_{x\to 2}(x+2) = 4$

Example $\lim_{x\to 1} \frac{x^3-1}{x-1} = \lim_{x\to 1} \frac{(x-1)(x^2+x+1)}{x-1} = \lim_{x\to 1}(x^2+x+1) = 3$

Example

$$\begin{aligned}
\lim_{x\to 0} \frac{\sqrt{x+1}-1}{x} &= \lim_{x\to 0} \frac{\sqrt{x+1}-1}{x} \cdot \frac{\sqrt{x+1}+1}{\sqrt{x+1}+1} \\
&= \lim_{x\to 1} \frac{x+1-1}{x(\sqrt{x+1}+1)} \\
&= \lim_{x\to 1} \frac{1}{\sqrt{x+1}+1} = \frac{1}{2}
\end{aligned}$$

Example $\lim_{x\to 2} \frac{x^2-4}{x+2} = \frac{0}{4} = 0$

Example Since $\lim_{x\to 2^+} \frac{x+2}{x-2} = \infty$ and $\lim_{x\to 2^-} \frac{x+2}{x-2} = -\infty$, all we can say about $\lim_{x\to 2} \frac{x+2}{x-2}$ is that it does not exist.

Example Suppose

$$f(x) = \begin{cases} x^2 - 1, & \text{if } x \le 0 \\ x + 4, & \text{if } x > 0. \end{cases}$$

Then

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-}(x^2 - 1) = -1$$

and

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} (x+4) = 4.$$

Hence $\lim_{x\to 0} f(x)$ does not exist.

Example Suppose

$$g(t) = \begin{cases} 3t+1, & \text{if } t \leq 1 \\ t^2+3, & \text{if } t > 1. \end{cases}$$

Then

$$\lim_{t\to 1^-} g(t) = \lim_{t\to 1^-} (3t+1) = 4$$

and

$$\lim_{t\to 1^+} g(t) = \lim_{t\to 1^+} (t^2+3) = 4.$$

Hence $\lim_{t\to 1} g(t) = 4$.

Example Let $f(x) = x\sin(\frac{1}{x})$. We have seen that $\lim_{x\to 0} \sin(\frac{1}{x})$ does not exist. However, since

$$-1 \leq \sin(\frac{1}{x}) \leq 1$$

for any value of x, we have

$$-x \leq x\sin(\frac{1}{x}) \leq x$$

whenever $x > 0$ and

$$-x \geq x\sin(\frac{1}{x}) \geq x$$

whenever $x < 0$. Now both $\lim_{x\to 0} x = 0$ and $\lim_{x\to 0}(-x) = 0$, so we must have

$$\lim_{x\to 0^+} x\sin(\frac{1}{x}) = 0$$

and

$$\lim_{x\to 0^-} x\sin(\frac{1}{x}) = 0.$$

Thus

$$\lim_{x\to 0} x\sin(\frac{1}{x}) = 0.$$

This is an example of the *squeeze theorem*.