Difference Equations to **D**ifferential Equations

Section 6.2

The Natural Logarithm Function

In the last example of Section 6.1 we saw the need for solving an equation of the form

$$e^x = b.$$

for x in terms of b. In general, for a given function f, a function g defined on the range of f is called the *inverse* of f if

$$g(f(x)) = x \tag{6.2.1}$$

for all x in the domain of f and

$$f(g(x)) = x \tag{6.2.2}$$

for all x in the domain of g. That is, if $f(x) = y$, then $g(y) = x$ and if $g(x) = y$, then $f(y) = x$. In order for a function f to have an inverse function g, for every point y in the range of f there must exist a unique point x in the domain of f such that $f(x) = y$, in which case $g(y) = x$. In other words, for any two points x_1 and x_2 in the domain of f, we must have $f(x_1) \neq f(x_2)$. Now this will be the case if f is increasing on its domain, since, for such an f, $x_1 < x_2$ implies $f(x_1) < f(x_2)$. In particular, a function f with domain (a, b) will have an inverse if $f'(x) > 0$ for all x in (a, b). Hence, since

$$\frac{d}{dx}e^x = e^x > 0$$

for all x in $(-\infty, \infty)$, the function $f(x) = e^x$ must have an inverse defined for every point in its range, namely, $(0, \infty)$. We call this inverse function the *natural logarithm function*.

Definition The inverse of the exponential function is called the *natural logarithm function*. The value of the natural logarithm function at a point x is denoted $\log(x)$.

Thus, by definition,

$$\log(e^x) = x \tag{6.2.3}$$

and

$$e^{\log(x)} = x. \tag{6.2.4}$$

Equivalently,

$$y = e^x \text{ if and only if } \log(y) = x. \tag{6.2.5}$$

Another common notation for $\log(x)$ is $\ln(x)$. In fact, most calculators use $\ln(x)$ for the natural logarithm of x and $\log(x)$ for the base 10 logarithm of x. However, since the natural logarithm function is the fundamental logarithm function of interest to us, we will denote it by $\log(x)$ and often refer to it simply as the logarithm function.

Example In the last example of Section 6.1 we needed to solve the equation

$$e^{10\alpha} = 1.114.$$

Taking the natural logarithm of both sides of this equation gives us

$$\log(e^{10\alpha}) = \log(1.114),$$

from which we obtain

$$10\alpha = \log(1.114),$$

and hence

$$\alpha = \frac{\log(1.114)}{10}.$$

Using a calculator and rounding to 4 decimal places, we find that $\alpha = 0.0108$.

Being the inverse of the exponential function, the logarithm function has domain $(0, \infty)$ (which is the range of the exponential function) and range $(-\infty, \infty)$ (which is the domain of the exponential function). Also, since $e^0 = 1$ and $e^1 = e$, it follows that $\log(1) = 0$ and $\log(e) = 1$.

Several basic algebraic properties of the logarithm function follow immediately from the algebraic properties of the exponential function. For example, since, for any positive numbers a and b,

$$e^{\log(a)+\log(b)} = e^{\log(a)}e^{\log(b)} = ab,$$

it follows, after taking the logarithm of both sides, that

$$\log(ab) = \log(a) + \log(b). \tag{6.2.6}$$

Similarly, since, for any positive numbers a and b,

$$e^{\log(a)-\log(b)} = e^{\log(a)}e^{-\log(b)} = \frac{e^{\log(a)}}{e^{\log(b)}} = \frac{a}{b},$$

we have

$$\log\left(\frac{a}{b}\right) = \log(a) - \log(b). \tag{6.2.7}$$

In particular,

$$\log\left(\frac{1}{b}\right) = \log(1) - \log(b) = -\log(b)$$

for any $b > 0$. Finally, if $a > 0$ and b is a rational number, then

$$e^{b\log(a)} = \left(e^{\log(a)}\right)^b = a^b$$

implies that

$$\log(a^b) = b\log(a). \tag{6.2.8}$$

We have restricted b to rational values here because we have not defined a^b for irrational values of b, except in the single case when $a = e$. However, the expression

$$e^{b\log(a)}$$

is defined for any value of b, rational or irrational. Hence the following definition provides a natural extension to the meaning of raising a number $a > 0$ to a power.

Definition If $a > 0$ and b is an irrational number, then we define

$$a^b = e^{b\log(a)}. \tag{6.2.9}$$

With this definition, we have

$$\log(a^b) = b\log(a). \tag{6.2.10}$$

for $a > 0$ and any value of b.

Example We now see that

$$2^\pi = e^{\pi\log(2)},$$

which, using a calculator, is 8.8250 to 4 decimal places.

The derivative of the logarithm function may be found using our knowledge of the derivative of the exponential function. Specifically, if $y = \log(x)$, then $e^y = x$. Thus, differentiating both sides of this expression with respect to x,

$$\frac{d}{dx}e^y = \frac{d}{dx}x,$$

from which we obtain

$$e^y\frac{dy}{dx} = 1.$$

Hence

$$\frac{dy}{dx} = \frac{1}{e^y} = \frac{1}{x}.$$

Since we started with $y = \log(x)$, this gives us the following proposition.

Proposition

$$\frac{d}{dx}\log(x) = \frac{1}{x}. \tag{6.2.11}$$

Example Combining the chain rule with the previous proposition, we have

$$\frac{d}{dx}\log(x^2+1) = \left(\frac{1}{x^2+1}\right)(2x) = \frac{2x}{x^2+1}.$$

Example It is worth noting that, in general, for any differentiable function f,

$$\frac{d}{dx}\log(f(x)) = \left(\frac{1}{f(x)}\right) f'(x) = \frac{f'(x)}{f(x)}. \tag{6.2.12}$$

Thus, for another example,

$$\frac{d}{dx}\log(3x^4+8) = \frac{12x^3}{3x^4+8}.$$

Example In some circumstances it is useful to use the properties of the logarithm function before attempting to differentiate. For example,

$$\begin{aligned}
\frac{d}{dx}\log\left(3x\sqrt{4x^2+2}\right) &= \frac{d}{dx}\left(\log(3x) + \log(4x^2+2)^{\frac{1}{2}}\right) \\
&= \frac{d}{dx}\left(\log(3) + \log(x) + \frac{1}{2}\log(4x^2+2)\right) \\
&= \frac{1}{x} + \frac{1}{2}\frac{8x}{4x^2+2} \\
&= \frac{1}{x} + \frac{2x}{2x^2+1}.
\end{aligned}$$

Turning to integrals, we note that

$$\frac{d}{dx}\log(x)) = \frac{1}{x}$$

implies that

$$\int \frac{1}{x}\,dx = \log(x) + c,$$

provided x is in the domain of the logarithm function, that is, $x > 0$. For $x < 0$, we have

$$\frac{d}{dx}\log|x| = \frac{d}{dx}\log(-x) = \frac{1}{-x}(-1) = \frac{1}{x},$$

showing that

$$\int \frac{1}{x}\,dx = \log|x| + c$$

for $x < 0$. Since $|x| = x$ when $x > 0$, we can combine the above results into one statement.

Proposition

$$\int \frac{1}{x}\,dx = \log|x| + c. \tag{6.2.13}$$

Example To evaluate $\displaystyle\int \frac{x}{x^2+1}\,dx$ we make the substitution

$$\begin{aligned}
u &= x^2+1 \\
du &= 2x dx.
\end{aligned}$$

Thus $\frac{1}{2}\,du = xdx$, so

$$\int \frac{x}{x^2+1}\,dx = \frac{1}{2}\int \frac{1}{u}\,du = \frac{1}{2}\log|u| + c = \frac{1}{2}\log(x^2+1) + c,$$

where we have removed the absolute value sign since $x^2 + 1 > 0$ for all x.

Example To evaluate

$$\int \tan(x)dx = \int \frac{\sin(x)}{\cos(x)}\,dx,$$

we make the substitution

$$\begin{aligned} u &= \cos(x) \\ du &= -\sin(x)dx \end{aligned}.$$

Thus $-du = \sin(x)dx$, so

$$\int \tan(x)dx = -\int \frac{1}{u}\,du = -\log|u| + c = \log|\cos(x)| + c.$$

Example To evaluate $\int \sec(x)dx$, we first multiply $\sec(x)$ by

$$\frac{\sec(x)+\tan(x)}{\sec(x)+\tan(x)}$$

to obtain

$$\int \sec(x)dx = \int \frac{\sec^2(x) + \sec(x)\tan(x)}{\sec(x)+\tan(x)}\,dx.$$

Then we make the substitution

$$\begin{aligned} u &= \sec(x) + \tan(x) \\ du &= (\sec(x)\tan(x) + \sec^2(x))dx. \end{aligned}$$

Hence

$$\int \sec(x)dx = \int \frac{1}{u}\,du = \log|u| + c = \log|\sec(x) + \tan(x)| + c.$$

Note that this example does not illustrate a general technique for evaluating integrals, but rather a nice trick that works in this specific case. In fact, it is just as easy to remember the value of the integral as it is to remember the trick that was used to find it.

Example We may evaluate $\int \log(x)dx$ using integration by parts. To do so, we choose

$$\begin{aligned} u &= \log(x) & dv &= dx \\ du &= \frac{1}{x}\,dx & v &= x. \end{aligned}$$

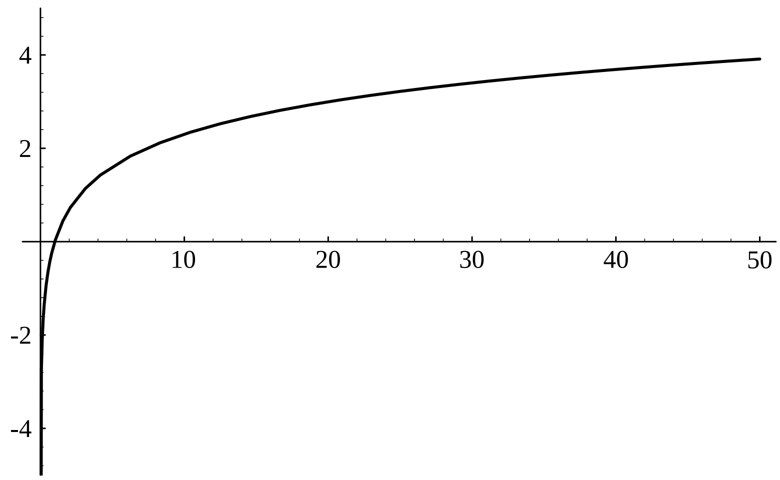

Figure 6.2.1 Graph of $y = \log(x)$

Then

$$\int \log(x)dx = x\log(x) - \int dx = x\log(x) + x + c.$$

We can now put together enough information to obtain a geometric understanding of the graph of $y = \log(x)$. Since

$$\frac{d}{dx}\log(x) = \frac{1}{x} > 0$$

for all x in $(0, \infty)$, the graph of $y = \log(x)$ is increasing on $(0, \infty)$. Note however that the slope of the graph decreases toward 0 as x increases; although the graph is always increasing, the rate of increase diminishes as x increases. This is also seen in the fact that

$$\frac{d^2}{dx^2}\log(x) = -\frac{1}{x^2} < 0$$

for all $x > 0$. As a consequence, the graph is concave down on $(0, \infty)$. Since the logarithm function is the inverse of the exponential function, it follows from

$$\lim_{x\to\infty} e^x = \infty$$

that

$$\lim_{x\to\infty} \log(x) = \infty \tag{6.2.14}$$

and from

$$\lim_{x\to-\infty} e^x = 0$$

that

$$\lim_{x\to 0^+} \log(x) = -\infty. \tag{6.2.15}$$

From (6.2.14) we see that, even though the slope of $y = \log(x)$ decreases toward 0 as x increases, y will continue to grow without any bound. From (6.2.15) we see that the y-axis is a vertical asymptote for the graph. Using this geometric information, we can understand why the graph of $y = \log(x)$ looks like it does in Figure 6.2.1. You should compare this graph with the graph of $y = e^x$ in Figure 6.1.1

We will use the relationship

$$\log(x) = \int_1^x \frac{1}{t}\, dt \tag{6.2.16}$$

to find the Taylor series representation for the logarithm function. Since, as we saw in Section 5.8,

$$\frac{1}{t} = \frac{1}{1-(1-t)} = \sum_{n=0}^{\infty}(1-t)^n = \sum_{n=0}^{\infty}(-1)^n(t-1)^n$$

for $0 < t < 2$, it follows that

$$\begin{aligned}
\log(x) &= \int_1^x \frac{1}{t}\, dt \\
&= \sum_{n=0}^{\infty}(-1)^n \int_1^x (t-1)^n dt \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n(x-1)^{n+1}}{n+1} \\
&= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x-1)^n}{n} \\
&= (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \cdots
\end{aligned}$$

for $0 < x < 2$. Hence

$$\log(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x-1)^n}{n} = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \cdots$$

is the Taylor series representation of $\log(x)$ on $(0, 2)$. Notice that at $x = 0$, this series becomes a multiple of the harmonic series, and so does not converge, while at $x = 2$, it is the alternating harmonic series, which does converge. Thus we would suspect that

$$\log(2) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots.$$

This is in fact true, and may be verified using Taylor's theorem (see Problem 6).

We will end this section by extending an old result. In Chapter 3 we saw that for any rational number $n \neq 0$,

$$\frac{d}{dx}x^n = nx^{n-1}.$$

Now that we have defined x^n for irrational n (provided $x > 0$), we see that

$$\frac{d}{dx}x^n = \frac{d}{dx}e^{n\log(x)} = \frac{n}{x}e^{n\log(x)} = \frac{nx^n}{x} = nx^{n-1}$$

for any real number $n \neq 0$.

Proposition For any real number $n \neq 0$,

$$\frac{d}{dx}x^n = nx^{n-1}. \tag{6.2.17}$$

Example Note that

$$\frac{d}{dx}x^\pi = \pi x^{\pi-1},$$

while

$$\frac{d}{dx}\pi^x = \frac{d}{dx}e^{x\log(\pi)} = \log(\pi)e^{x\log(\pi)} = \log(\pi)\pi^x.$$

In Section 6.3 we will consider some applications of the exponential and logarithm functions.

Problems

1. Let $a = \log(2)$ and $b = \log(3)$. Find the following in terms of a and b.

   (a) $\log(6)$ (b) $\log(1.5)$
   (c) $\log(9)$ (d) $\log(12)$

2. Find the derivative of each of the following functions.

   (a) $f(x) = \log(3x^2)$ (b) $g(t) = t^3\log(3t+4)$
   (c) $g(x) = \log\left(4x^2\sqrt{x^2+5}\right)$ (d) $h(t) = \log\left(\dfrac{13t^2+1}{5t+3}\right)$
   (e) $f(x) = e^{2x}\log(5x)$ (f) $g(z) = 3z\log(4z+5)$
   (g) $h(x) = \log(\log(x))$ (h) $f(x) = 2^x$
   (i) $f(x) = x^e$ (j) $f(t) = \log\sqrt{\dfrac{4t^2+3}{t^4+1}}$

3. Evaluate each of the following integrals.

   (a) $\displaystyle\int \frac{1}{2x}\,dx$ (b) $\displaystyle\int \frac{3x}{x^2+2}\,dx$
   (c) $\displaystyle\int \frac{5x}{3x^2+1}\,dx$ (d) $\displaystyle\int \frac{3x^2}{4x^3+15}\,dx$
   (e) $\displaystyle\int \tan(3x)dx$ (f) $\displaystyle\int \cot(x)dx$
   (g) $\displaystyle\int \csc(x)dx$ (h) $\displaystyle\int \frac{e^x-e^{-x}}{e^x+e^{-x}}\,dx$

4. Evaluate each of the following integrals.

   (a) $\displaystyle\int \log(3x)dx$ (b) $\displaystyle\int x\log(x)dx$

(c) $\displaystyle\int \frac{\log(x)}{x}\,dx$

(d) $\displaystyle\int 3x^2 \log(x)dx$

(e) $\displaystyle\int \log(x+1)dx$

(f) $\displaystyle\int (x^2+3)\log(x)dx$

(g) $\displaystyle\int \frac{1}{x\log(x)}\,dx$

(h) $\int 2^x dx$

5. (a) Show that
$$\lim_{x\to\infty} \frac{\log(x)}{x} = 0.$$
What does this say about the rate of growth of $\log(x)$ as x increases?

   (b) Show that for any real number $p > 0$,
$$\lim_{x\to\infty} \frac{\log(x)}{x^p} = 0.$$
What does this say about the rate of growth of $\log(x)$ as x increases?

6. (a) Use Taylor's theorem to show that the alternating harmonic series converges to $\log(2)$. That is, if $r_n(x)$ is the error in approximating $\log(x)$ by the nth order Taylor polynomial at x, show that $\lim\limits_{n\to\infty} r_n(2) = 0$.

   (b) Use the Taylor series for $\log(x)$ about 1 to approximate $\log(2)$ with an error of less than 0.005.

   (c) Use the Taylor series for $\log(x)$ about 1 to approximate $\log(1.5)$ with an error of less than 0.001.

7. Graph each of the following on the given interval.

   (a) $y = \log(3x)$ on $(0, 20]$

   (b) $y = \log|x|$ on $[20, 20]$

   (c) $x = \dfrac{\log(t)}{t}$ on $(0, 10]$

   (d) $x = t^2\log(t)$ on $(0, 3]$

   (e) $y = \sin(\log(\theta))$ on $(0, 2]$

   (f) $y = \log(x^2)$ on $(0, 50]$

8. Compare the graphs of $y = 2^x$ and $y = \left(\dfrac{1}{2}\right)^x$.

9. Use the integral test to show that the harmonic series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

10. (a) Find $\lim\limits_{x\to\infty} \log(\log(x))$.

    (b) Show that
$$\lim_{x\to\infty} \frac{\log(\log(x))}{\log(x)} = 0.$$
What does this say about the rate of growth of $\log(\log(x))$ as x increases?

    (c) Graph $y = \log(\log(x))$.

(d) Find the value of x such that $\log(x) = 20$.

(e) Find the value of x such that $\log(\log(x)) = 20$.

11. Find the length of the curve $y = \log(x)$ over the interval $[1, 10]$.

12. Suppose x is a function with $\dot{x}(t) = \alpha x(t)$, $x(0) = 100$, and $x(5) = 200$. Find $x(t)$.

13. Given that g is the inverse function of f, show that

$$g'(x) = \frac{1}{f'(g(x))}.$$

Use this result to show that $\dfrac{d}{dx}\log(x)) = \dfrac{1}{x}$.