**Section 3.8**

Finding Maximum and Minimum Values

Problems involving finding the maximum or minimum value of a quantity occur frequently in mathematics and in the applications of mathematics. A company may want to maximize its profit or minimize its costs; a farmer may want to maximize the yield from his crop or minimize the amount of irrigation equipment needed to water his fields; an airline may want to maximize its fuel efficiency or minimize the length of its routes. Methods for solving some optimization problems are so computationally intense that they challenge, and sometimes even go beyond, the fastest computers currently available. An example of such a problem is the famous traveling salesman problem, in which a salesman wishes to visit a certain set of cities using the shortest possible route. In this section we will not consider problems of this type, but rather we will confine ourselves to problems involving continuous functions of a single independent variable.

Closed intervals

We will start with the simplest case. Suppose f is a continuous function on a closed interval $[a, b]$. From the Extreme Value Theorem we know that f attains both a maximum value and a minimum value on the interval. We now look for candidates at which these values might occur. To start, an extreme value could occur at one of the endpoints. For example, the maximum value of $f(x) = x^2$ on $[0, 1]$ occurs at $x = 1$. If an extreme value occurs in the open interval (a, b) at a point c where f is differentiable, then f has a local extremum at c and so, from our work in Section 3.7, we know that $f'(c) = 0$. For example, the minimum value of $f(x) = x^2$ on $[-1, 1]$ occurs at $x = 0$ and $f'(0) = 0$. Finally, the only other candidates for the locations of extreme values would be points where f' is undefined. For example, the minimum value of $f(x) = |x|$ on $[-1, 1]$ occurs at $x = 0$, where f' is not defined. Hence we are led to the following conclusion: The extreme values of a continuous function f on a closed interval are located either at the endpoints of the interval, at points where f' is 0, or at points where f' is undefined. The following terminology will help us state this more easily.

Definition   If f is differentiable at c and $f'(c) = 0$, then we call c a *critical point* or *stationary point* of f. A point c at which the derivative of f is not defined is called a *singular point* of f.

Thus we know that the candidates for the location of the extreme values of a continuous function on a closed interval fall into three categories: (a) endpoints of the interval, (b) critical points, and (c) singular points. To determine the extreme values of such a function f, we identify all these points, evaluate f at each one, and identify the largest and smallest values.
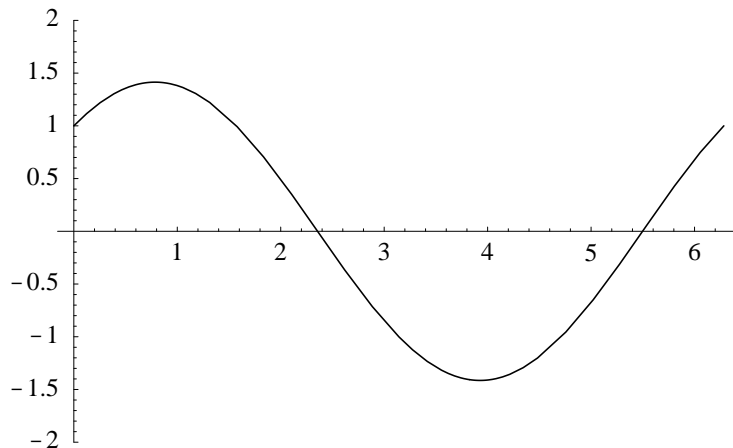
Figure 3.8.1 Graph of $g(t) = \cos(t) + \sin(t)$ on $[0, 2\pi]$

Example   Suppose we wish to find the maximum and minimum values of

$$g(t) = \cos(t) + \sin(t)$$

on the interval $[0, 2\pi]$. Then

$$g'(t) = -\sin(t) + \cos(t),$$

so $g'(t) = 0$ when

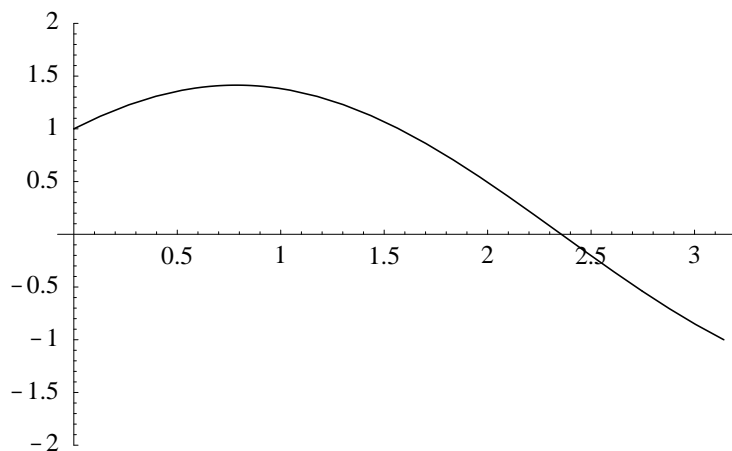$$\cos(t) = \sin(t). \tag{3.8.1}$$

Now $\cos(t)$ and $\sin(t)$ are never simultaneously 0, so we may divide both sides of (3.8.1) by $\cos(t)$ to see that $g'(t) = 0$ when

$$\tan(t) = 1.$$

Considering only the interval $[0, 2\pi]$, this implies $t = \frac{\pi}{4}$ or $t = \frac{5\pi}{4}$. Since there are no singular points, we evaluate g at the endpoints and at the critical points:

$$g(0) = 1$$
$$g\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \sqrt{2}$$
$$g\left(\frac{5\pi}{4}\right) = -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} = -\sqrt{2}$$
$$g(2\pi) = 1$$

Thus g has a maximum value of $\sqrt{2}$ at $t = \frac{\pi}{4}$ and a minimum value of $-\sqrt{2}$ at $t = \frac{5\pi}{4}$. See Figure 3.8.1.

Figure 3.8.2 Graph of $g(t) = \cos(t) + \sin(t)$ on $[0, \pi]$

Example Note that if the interval in the previous example had been $[0, \pi]$, then the only critical point would be $\frac{\pi}{4}$. In this case we would evaluate:

$$g(0) = 1$$
$$g\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \sqrt{2}$$
$$g(\pi) = -1$$

Hence the maximum value of g on $[0, \pi]$ is, as before, $\sqrt{2}$ at $x = \frac{\pi}{4}$, but the minimum value of g on $[0, \pi]$ is -1 at $t = \pi$. See Figure 3.8.2.

Example Consider the function $f(x) = x^{\frac{2}{3}}$ on the interval $[-1, 1]$. Then

$$f'(x) = \frac{2}{3}x^{-\frac{1}{3}} = \frac{2}{x^{\frac{1}{3}}},$$

which is never 0, but is undefined at 0. Thus f has a singular point at 0, but no critical points in $[-1, 1]$. To find the extreme values of f, we evaluate:

$$f(-1) = 1$$
$$f(0) = 0$$
$$f(1) = 1$$

Hence f has a minimum value of 0 at $x = 0$ and a maximum value of 1 which occurs at both $x = -1$ and $x = 1$. See Figure 3.8.3.

Example A quality control engineer wishes to determine the probability that a certain type of light bulb will fail within 1000 hours of use. To do so, she tests 100 such light bulbs for 1000 hours each and finds that 20 of them failed within that time period. If p is the
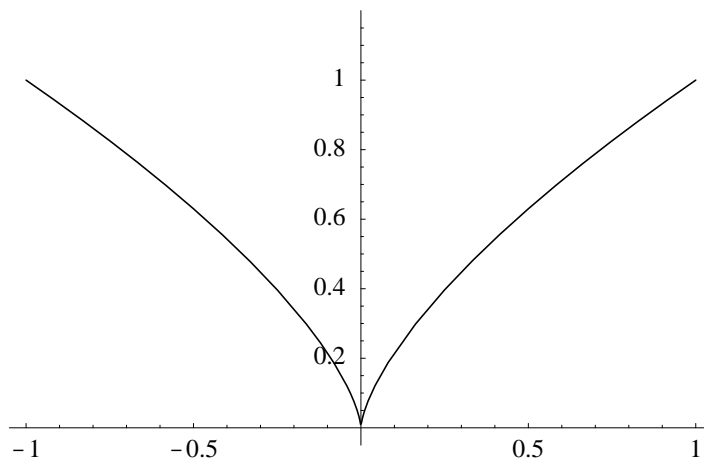
Figure 3.8.3 Graph of $f(x) = x^{\frac{2}{3}}$ on $[-1, 1]$

probability that a single light bulb fails within 1000 hours, the probability of the observed sequence of successes and failures is given by

$$L(p) = p^{20}(1-p)^{80},$$

with $0 \leq p \leq 1$. Note here that $1 - p$ is the probability that a single light bulb does not fail in the 1000 hour test. If we think of p as representing the proportion of all such light bulbs that will fail within 1000 hours, then $L(p)$ represents the proportion of times that a sequence of 100 tests will yield the observed sequence of successes and failures. The quality control engineer would like to use this information to estimate p, the true probability of failure for this particular type of light bulb. One common procedure is to estimate p by the value $\hat{p}$ which maximizes the probability of the observed sequence; that is, $\hat{p}$ is the value of p that makes the given observations most likely to occur. Hence we want to maximize the function L on the interval $[0, 1]$. To find the critical points, we compute

$$\begin{aligned}
L'(p) &= p^{20}(80(1-p)^{79}(-1)) + (1-p)^{80}(20p^{19}) \\
&= -80p^{20}(1-p)^{79} + 20p^{19}(1-p)^{80} \\
&= 20p^{19}(1-p)^{79}(-4p + (1-p)) \\
&= 20p^{19}(1-p)^{79}(1-5p).
\end{aligned}$$

Thus $L(p) = 0$ when $p = 0$, $p = 0.2$, or $p = 1$, and so the only critical point in the interval $(0, 1)$ is 0.2. Evaluating L at the endpoints and at the critical point yields:

$$\begin{aligned}
L(0) &= 0 \\
L(0.2) &= (0.2)^{20}(0.8)^{80} = 1.853 \times 10^{-22} \\
L(1) &= 0
\end{aligned}$$

Thus the quality control engineer would take the value $\hat{p} = 0.2$ as her estimate of the probability that this type of light bulb will fail within the first 1000 hours of use. See Figure 3.8.4.
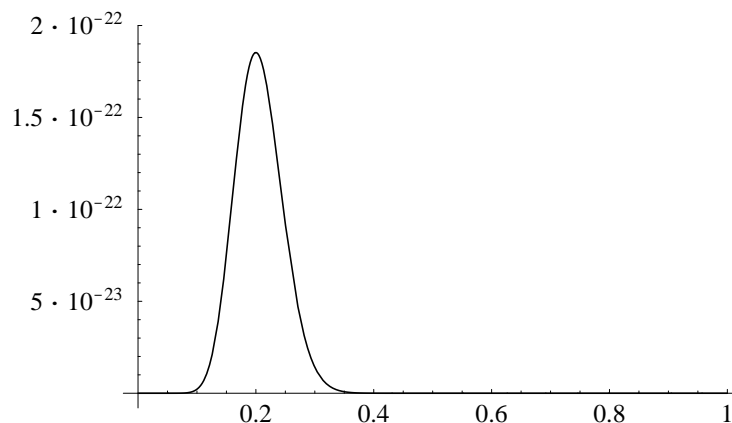
Figure 3.8.4 Graph of $L(p) = p^{20}(1-p)^{80}$ on $[0,1]$

The answer in the previous example should not be surprising: Given that 20 out of 100 light bulbs in the sample failed within 1000 hours, it seems evident that our best guess for the probability that a randomly chosen light bulb manufactured by this process will fail in less than 1000 hours is $\frac{20}{100} = \frac{1}{5}$. However, our example illustrates a general technique, called *maximum likelihood estimation*, which is widely used in applications to estimate statistical parameters. Moreover, there are other methods which would yield different answers, one popular alternative being $\frac{21}{102}$.

Open intervals

Finding the extreme values of a continuous function f on an interval I which is not closed introduces some new problems. Foremost among these is that there is no guarantee, like the Extreme Value Theorem, that extreme values even exist. However, the following special case arises frequently in practice and may be handled routinely. Suppose $I = (a, b)$ is an open interval, f and f' are continuous on I, f has a critical point at a point c in I which has been determined to be the location of a local minimum, and f has no other critical points in I. Then, by the Intermediate Value Theorem, f' cannot change sign on (a, c). Hence, since there is a local minimum at c, $f'(x) < 0$ for all x in (a, c). Similarly, we must have $f'(x) > 0$ for all x in (c, b). Thus f is decreasing on (a, c) and increasing on (c, b), so $f(c)$ must be the minimum value of f on (a, b). An analogous argument shows that if, under these conditions, f has a local maximum at c, then $f(c)$ is the maximum value of f on (a, b). The following proposition summarizes these statements.

Proposition   Suppose f and f' are continuous on an open interval (a, b) and c is the only critical point of f in (a, b). If f has a local minimum at c, then $f(c)$ is the minimum value of f on (a, b); if f has a local maximum at c, then $f(c)$ is the maximum value of f on (a, b).

Of course, to make use of the proposition, we must first have a method for determining the location of local extreme values. Given a differentiable function f on an open interval I, we know that the local extreme values will occur only at critical points; hence, the first step in determining the location of local extreme values is to find all the critical points of
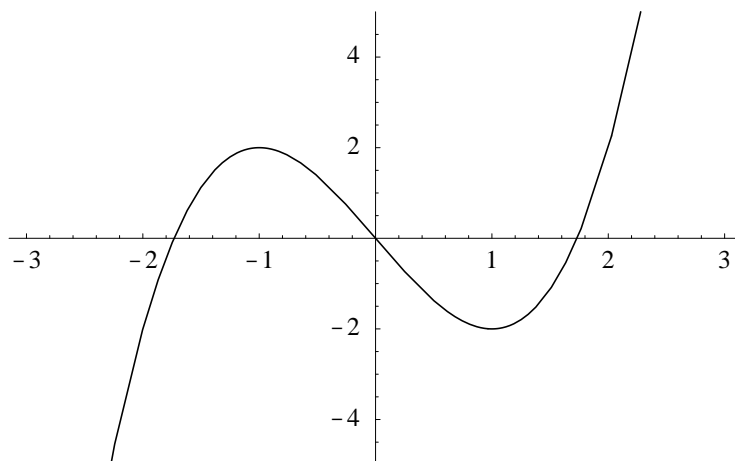
Figure 3.8.5 Graph of $f(x) = x^3 - 3x$

f in I. A given critical point c may then be classified as the location of a local maximum, a local minimum, or neither by examining the behavior of f' on either side of c. That is, if f' is negative to the left of c and positive to the right of c, then f is decreasing before c and increasing after c, making c the location of a local minimum. Conversely, if f' is positive to the left of c and negative to the right of c, then f is increasing before c and decreasing after c, making c the location of a local maximum. If f' is either negative on both sides of c or positive on both sides of c, then f has neither a local minimum nor a local maximum at c. The procedure just described is sometimes referred to as the *first derivative test for local extrema*.

Example   Suppose $f(x) = x^3 - 3x$. Then

$$f'(x) = 3x^2 - 3 = 3(x^2 - 1) = 3(x - 1)(x + 1),$$

so $f'(x) = 0$ when $x = -1$ or $x = 1$. Now $x^2 - 1 < 0$ when $-1 < x < 1$ and $x^2 - 1 > 0$ for all other values of x. Hence $f'(x) > 0$ when $x < -1$ or $x > 1$, and $f'(x) < 0$ when $-1 < x < 1$. Hence f is increasing on $(-\infty, -1)$ and on $(1, \infty)$, and f is decreasing on $(-1, 1)$. Thus f changes from increasing to decreasing at $x = -1$, implying that f has a local maximum at this point, and f changes from decreasing to increasing at $x = 1$, implying that f has a local minimum at this point. Since $f(-1) = 2$ and $f(1) = -2$, we conclude that f has a local maximum of 2 at $x = -1$ and a local minimum of -2 at $x = 1$. Note that this verifies the claim we made in Section 3.7 after looking at the graph of f (which is repeated in Figure 3.8.5).

Our next step will require the introduction of the derivative of f', called the *second derivative* of f, and denoted f''.

Example   If $f(x) = x^3 - 3x^2$, then

$$f'(x) = 3x^2 - 6x$$

and

$$f''(x) = 6x - 6.$$

In Leibniz notation, if $y = f(x)$, then $f''(x)$ is denoted by $\frac{d^2y}{dx^2}$, which may be thought of as

$$\frac{d^2}{dx^2}y = \frac{d}{dx}\left(\frac{d}{dx}y\right).$$

In Newton's notation, if $x = f(t)$, then $\ddot{x} = f''(t)$.

Example   If $y = x\sin(3x)$, then

$$\frac{dy}{dx} = 3x\cos(3x) + \sin(3x),$$

so

$$\begin{aligned}
\frac{d^2y}{dx^2} &= \frac{d}{dx}(3x\cos(3x) + \sin(3x)) \\
&= -9x\sin(3x) + 3\cos(3x) + 3\cos(3x) \\
&= -9x\sin(3x) + 6\cos(3x).
\end{aligned}$$

Example   If $x = 3t^6 - 2\cos(6t)$, then

$$\dot{x} = 18t^5 + 12\sin(6t)$$

and

$$\ddot{x} = 90t^4 + 72\cos(6t).$$

Now suppose f, f', and f'' are all continuous on an interval (a, b) containing a critical point c. Moreover, suppose $f''(c) < 0$. Since f'' is continuous, this assumption in fact implies that $f''(c) < 0$ on some open interval about c, and hence that f' is a decreasing function on some open interval about c. But $f'(c) = 0$, so for f' to be decreasing it must be the case that f' is positive to the left of c and negative to the right of c. This means, by the first derivative test discussed above, that f must have a local maximum at c. Similarly, if $f''(c) > 0$, then f' is negative to the left of c and positive to the right of c, showing that f has a local minimum at c. This important result is known as the *second derivative test for local extrema*

Second Derivative Test   Suppose f, f', and f'' are all continuous on an open interval (a, b) and that c is critical point of f in (a, b). Then f has a local maximum at c if $f''(c) < 0$ and f has a local minimum at c if $f''(c) > 0$.

Example   Consider the function $g(x) = \dfrac{x}{1 + x^2}$. Then

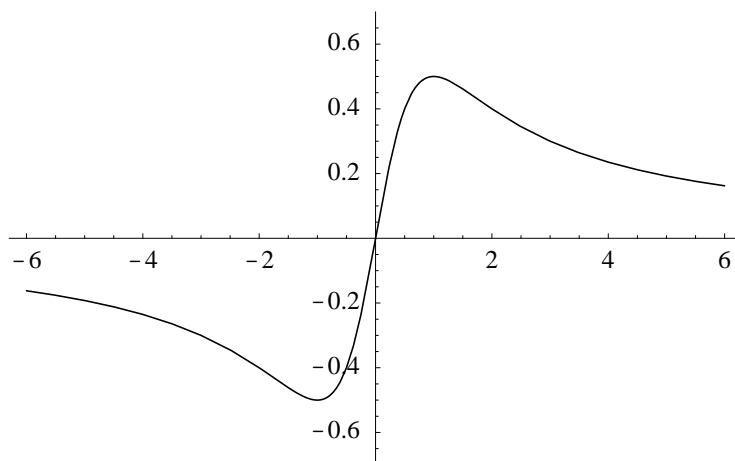$$g'(x) = \frac{(1 + x^2)(1) - (x)(2x)}{(1 + x^2)^2} = \frac{1 - x^2}{(1 + x^2)^2},$$

Figure 3.8.6 Graph of $g(x) = \dfrac{x}{1 + x^2}$

so $g'(x) = 0$ when $1 - x^2 = 0$. Thus g has two critical points, $x = -1$ and $x = 1$. Now

$$
\begin{aligned}
g''(x) &= \frac{(1 + x^2)^2(-2x) - (1 - x^2)(2(1 + x^2)(2x))}{(1 + x^2)^4} \\
&= \frac{-2x(1 + x^2) - 4x(1 - x^2)}{(1 + x^2)^3} \\
&= \frac{-6x + 2x^3}{(1 + x^2)^3} \\
&= \frac{2x(x^2 - 3)}{(1 + x^2)^3}.
\end{aligned}
$$

Thus $g''(-1) = 0.5 > 0$ and $g(1) = -1 < 0$, implying that g has a local minimum at $x = -1$ and a local maximum at $x = 1$. Since $g(-1) = -0.5$ and $g(1) = 0.5$, we conclude that g has a local minimum of -0.5 at $x = -1$ and a local maximum of 0.5 at $x = 1$.

Using the facts that the critical points of g are -1 and 1 and that g has a local minimum at $x = -1$, we may conclude that g must be decreasing on $(-\infty, -1)$ and increasing on $(-1, 1)$. Similarly, g having a local maximum at 1 implies that g must be increasing on $(-1, 1)$ and decreasing on $(1, \infty)$. Moreover,
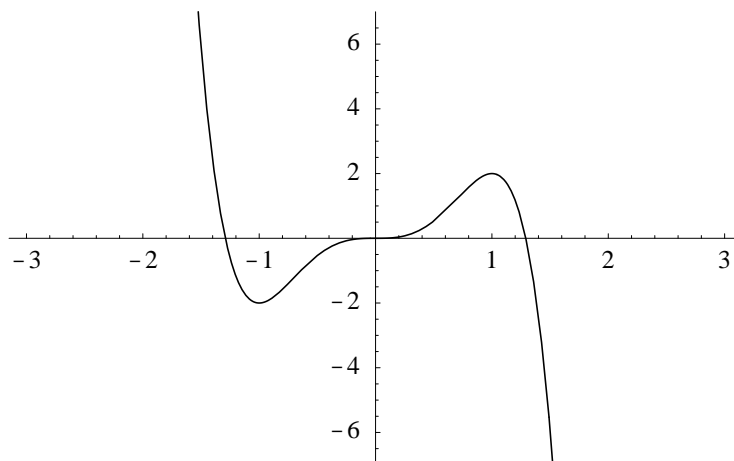
$$
\lim_{x \to -\infty} g(x) = \lim_{x \to -\infty} \frac{x}{1 + x^2} = \lim_{x \to -\infty} \frac{\dfrac{1}{x}}{\dfrac{1}{x^2} + 1} = \frac{0}{1} = 0
$$

and

$$
\lim_{x \to \infty} g(x) = \lim_{x \to \infty} \frac{x}{1 + x^2} = \lim_{x \to \infty} \frac{\dfrac{1}{x}}{\dfrac{1}{x^2} + 1} = \frac{0}{1} = 0,
$$

showing that the x-axis is a horizontal asymptote for the graph of g. Putting these observations together, we can see why the graph of g looks as it does in Figure 3.8.6.

Figure 3.8.7 Graph of $f(x) = 5x^3 - 3x^5$

Example   Suppose $f(x) = 5x^3 - 3x^5$. Then

$$f'(x) = 15x^2 - 15x^4 = 15x^2(1 - x^2) = 15x^2(1 - x)(1 + x),$$

from which we see that the critical points of f are -1, 0, and 1. Now

$$f''(x) = 30x - 60x^3,$$

so $f''(-1) = 30 > 0$, $f''(0) = 0$, and $f''(1) = -30 < 0$. Thus f has a local minimum of -2 at $x = -1$ and a local maximum of 2 at $x = 1$. Unfortunately, $f''(0) = 0$, so the second derivative test gives us no information about the nature of the critical point 0. However, since f has a local minimum at $x = -1$, f must be decreasing on $(-\infty, -1)$ and increasing on $(-1, 0)$; moreover, since f has a local maximum at $x = 1$, f must be increasing on $(0, 1)$ and decreasing on $(1, \infty)$. Thus f is increasing on both $(-1, 0)$ and $(1, 0)$, from which we conclude that f has neither a local minimum nor a local maximum at $x = 0$. If we add in that $f(0) = 0$,

$$\lim_{x \to -\infty} f(x) = \lim_{x \to -\infty} x^5 \left(\frac{5}{x^2} - 3 \right) = \infty,$$

and

$$\lim_{x \to \infty} f(x) = \lim_{x \to \infty} x^5 \left(\frac{5}{x^2} - 3 \right) = -\infty,$$

we can see why the graph of f look as it does in Figure 3.8.7.

It is worth emphasizing that, for a critical point c, $f''(c) = 0$ gives us no information about the nature of the critical point. The point may be the location of a local minimum, as 0 is for $f(x) = x^4$; a local maximum, as 0 is for $f(x) = -x^4$; or neither, as 0 is in the previous example. Thus, if $f''(c) = 0$ for a critical point c, the second derivative test is not applicable and some other method, such as the first derivative test, must be used to determine the nature of the point.
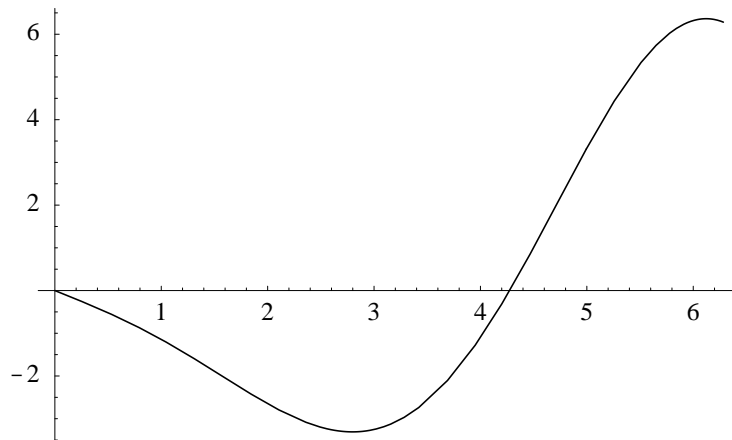
Figure 3.8.8 Graph of $g(t) = t\cos(t) - 2\sin(t)$

Now that we have techniques for determining the location of local minimums and maximums, we can return to our original problem of determining the maximum or minimum value of a function on an open interval.

Example   Suppose we want to find the minimum value of

$$f(t) = \frac{\sin(t)}{t^2}$$

on the interval $(0, 2\pi)$. Then

$$f'(t) = \frac{t^2\cos(t) - \sin(t)(2t)}{t^4} = \frac{t\cos(t) - 2\sin(t)}{t^3},$$

so $f'(t) = 0$ when $t\cos(t) - 2\sin(t) = 0$. We cannot solve this equation exactly, but, from the graph of $g(t) = t\cos(t) - 2\sin(t)$ in Figure 3.8.8, we can see that it has only one solution in the interval $(0, 2\pi)$. Applying Newton's method with initial guess $t_0 = 4$, we obtain the approximation 4.2748, to four decimal places. Hence f has exactly one critical point in $(0, 2\pi)$, namely, 4.2748. Now

$$f''(t) = \frac{t^3(-t\sin(t) + \cos(t) - 2\cos(t)) - (t\cos(t) - 2\sin(t))(3t^2)}{t^6}$$
$$= \frac{-t^4\sin(t) - 4t^3\cos(t) + 6t^2\sin(t)}{t^6}$$
$$= \frac{6\sin(t) - 4t\cos(t) - t^2\sin(t)}{t^4},$$

so $f''(4.2748) = 0.05499 > 0$. Thus f has a local minimum at $t = 4.2748$. Moreover, since 4.2748 is the only critical point in $(0, 2\pi)$ and $f(4.2748) = -0.04957$, we may conclude that the minimum value of f on $(0, 2\pi)$ is -0.04957, and this value occurs at $t = 4.2748$. See Figure 3.8.9.
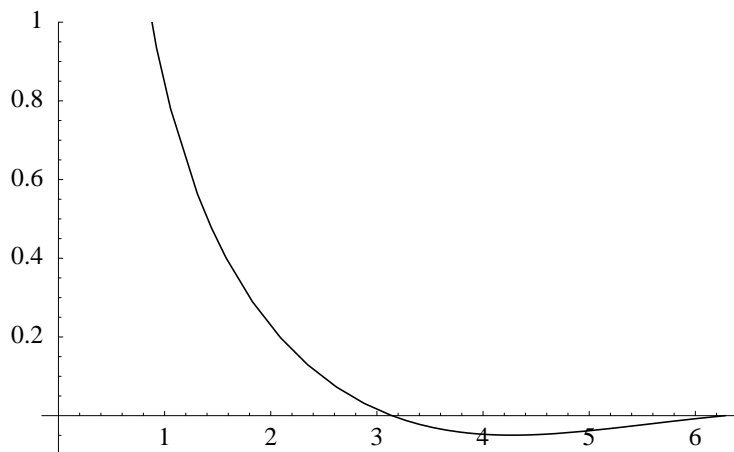
Figure 3.8.9 Graph of $f(t) = \dfrac{\sin(t)}{t^2}$

Example   A company wishes to produce a metal can in the shape of a right circular cylinder which minimizes the amount of metal needed in its construction, yet will have a volume of V cubic centimeters. If we denote the radius of the base of the can by r, the height of the can by h, and the surface area of the can by S, then

$$S = 2\pi r^2 + 2\pi rh, \tag{3.8.2}$$

where the first term represents the combined area of the base and the top of the can and the second term represents the area of the side of the can, which, when flattened out, is a rectangle of length $2\pi r$ (the circumference of the base of the can) and width h. Our goal is to find the values of r and h which minimize S subject to the constraint that the can has to hold a volume V. This constraint translates into the condition

$$V = \pi r^2 h,$$

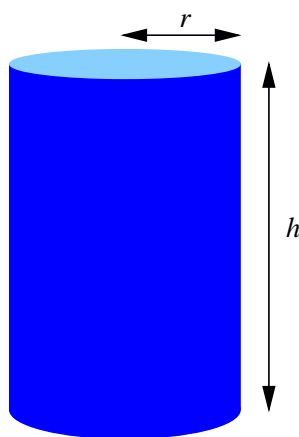which means we must have

$$h = \frac{V}{\pi r^2}. \tag{3.8.3}$$

Figure 3.8.10 A cylindrical can

Substituting the value of h in (3.8.3) into (3.8.2), we have

$$S = 2\pi r^2 + \frac{2V}{r}, \tag{3.8.4}$$

giving S as a function of r which we want to minimize on the interval $(0, \infty)$. Now

$$\frac{dS}{dr} = 4\pi r - \frac{2V}{r^2},$$

so $\frac{dS}{dr} = 0$ when

$$4\pi r = \frac{2V}{r^2},$$

that is, when

$$r^3 = \frac{V}{2\pi}.$$

Hence

$$r = \sqrt[3]{\frac{V}{2\pi}}$$

is the only critical point of S in $(0, \infty)$. Moreover,

$$\frac{d^2 S}{dr^2} = 4\pi + \frac{4V}{r^3},$$

so

$$\left. \frac{d^2 S}{dr^2} \right|_{r = \sqrt[3]{\frac{V}{2\pi}}} = 4\pi + 8\pi = 12\pi > 0.$$

Hence S has a local minimum at

$$r = \sqrt[3]{\frac{V}{2\pi}}.$$

Since this is the only critical point in $(0, \infty)$, this is in fact the location of the minimum value of S. Now for

$$r = \sqrt[3]{\frac{V}{2\pi}},$$

we have, using (3.8.3),

$$h = \frac{V}{\pi r^2} = \frac{V}{\pi \left(\frac{V}{2\pi} \right)^{\frac{2}{3}}} = \frac{2^{\frac{2}{3}} \pi^{\frac{2}{3}} V}{\pi V^{\frac{2}{3}}} = 2\sqrt[3]{\frac{V}{2\pi}} = 2r.$$

That is, the surface area of the can, and hence the amount of metal used in the can, is minimized when the height of the can is equal to the diameter of the can. Figure 3.8.11
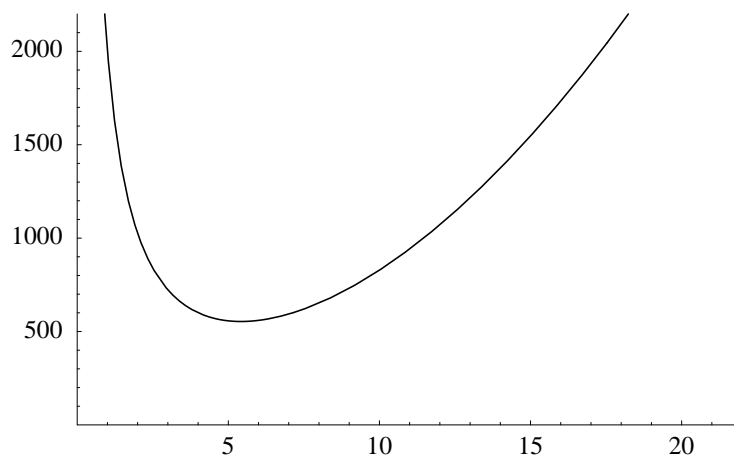
Figure 3.8.11 Graph of $S = 2\pi r^2 + \dfrac{2000}{r}$

shows the graph of S in the case $V = 1000$ cubic centimeters, in which case the surface area is minimized when r is approximately 5.42 centimeters.

Problems

1. Find the minimum and maximum values, and their locations, for each of the following functions on the given intervals.

   (a) $f(x) = x^2 - 4$ on $[-3, 4]$ (b) $f(x) = x^3 - 3x$ on $[-2, 4]$

   (c) $g(t) = \cos(t) - \sin(t)$ on $[-\pi, \pi]$ (d) $f(t) = 2t^3 + 3t^2 - 36t$ on $[-4, 3]$

   (e) $g(x) = x^2 \cos(x)$ on $[-2, 2]$ (f) $f(t) = \cos(t) + \sin(2t)$ on $[0, \pi]$

   (g) $f(t) = t^2 \sin(t)$ on $[0, \pi]$

2. A farmer wishes to fence in a rectangular field with 600 yards of fencing. What should the dimensions of the field be in order to maximize the area of the field?

3. A farmer wishes to fence in a rectangular field, using a straight river for one side, with 500 yards of fencing. What should the dimensions of the field be in order to maximize the area of the field?

4. Suppose the farmer in Problem 2 wishes to divide his field into two equal rectangular fields using a fence parallel to two of the sides. What should the dimensions of the field be in order to maximize the combined areas of the fields?

5. When a potter sells his pots for p dollars apiece, he can sell $D(p) = 2500 - p^2$ of them. Suppose the pots cost him $6.00 apiece to make. What price should the potter charge in order to maximize his profit?

6. A wire of unit length is to be cut into two pieces. One of the pieces will be used to form a square, the other a circle. Where should the wire be cut in order to maximum the total area enclosed by the square and the circle? Where should it be cut in order to minimize the total area enclosed by the square and the circle?

7. Find all local maximums and minimums, and their locations, for the following functions.

(a) $f(x) = 3x^2 + 5$

(b) $f(t) = t^4 + 3t^2$

(c) $g(t) = t^3 + 3t^2$

(d) $g(x) = \sin(x)\cos(x)$

(e) $f(x) = \dfrac{1}{1 + x^2}$

(f) $h(x) = \dfrac{x^2}{1 + x^2}$

(g) $g(x) = x^5 - x^3$

(h) $f(t) = t^4 - 2t^3$

(i) $g(t) = 3t^5 - 5t^4$

(j) $f(x) = \dfrac{x}{1 + 3x^2}$

8. Find the second derivative of each of the following functions.

(a) $f(x) = 3x^2 + 2x - 3$

(b) $g(t) = 13t^4 - 3t^3 + t^2 - 45$

(c) $s = \dfrac{1}{2t - 1}$

(d) $g(x) = \sin^2(3x)$

(e) $x = \sin(2t)\cos(4t)$

(f) $y = x^2\tan^2(3x)$

9. Find the maximum value of

$$f(x) = \frac{x - 1}{x^2}$$

on the interval $(0, \infty)$. Does f have a minimum value on $(0, \infty)$?

10. We found the minimum value of

$$f(t) = \frac{\sin(t)}{t^2}$$

on $(0, 2\pi)$ in an example. Does f have a maximum value on $(0, 2\pi)$?

11. A farmer wishes to construct a rectangular storage bin with a volume of 1000 cubic feet. Both the top and the bottom of the bin are to be squares. Find the dimensions of the bin which will minimize its surface area.

12. Suppose the bin in Problem 11 does not require a bottom. Find the dimensions of the bin which minimize surface area in this case.

13. Suppose the material for the top and the bottom of the bin in Problem 11 costs $2.00 per square foot while the material for the sides costs $3.00 per square foot. Find the dimensions of the bin which minimize its cost.

14. A metal can in the shape of a right circular cylinder without a top is to be made so that it holds 100 cubic centimeters. Find the dimensions of the can which minimize its surface area.

15. Suppose the material for the top and bottom of a can in the shape of a right circular cylinder costs $0.04 per square centimeter and the material for the side costs $0.02 per square centimeter. If the can must hold 1000 cubic centimeters, for what dimensions is the cost of the can minimized?

16. A metal can in the shape of a right circular cylinder is to be made so that it holds 500 cubic centimeters. Suppose the top and bottom of the can are cut from square pieces of metal, with the scraps being discarded afterwards. Assuming there is no waste in making the side of the can, find the dimensions of the can which minimize the amount of material needed to make it.

17. Show that the rectangle with maximum area for a given perimeter P is a square.

18. Show that the rectangle with minimum perimeter for a given area A is a square.

19. A quality control engineer is studying the failure rate of a certain type of beam under stress. Test beams are put under a steady stress for 1000 hours. If p is the probability that the beam passes the test, then $1 - p$ is the probability that the beam fails the test. Suppose that in 50 trials, only 5 beams fail the test.

    (a) If $L(p)$ is the probability of the observed sequence of successes and failures, explain why
    $$L(p) = p^{45}(1 - p)^5$$
    for $0 \leq p \leq 1$.

    (b) If $\hat{p}$ is the value of p which maximizes $L(p)$ on $[0, 1]$, show that $\hat{p} = 0.9$.

20. According to genetic theory, if a parent provides the gene A with probability θ and the gene a with probability $1 - \theta$, then the offspring is of genotype AA with probability θ^2, of genotype Aa with probability $2\theta(1 - \theta)$, and of genotype aa with probability $(1 - \theta)^2$. Suppose that in a sample of 100 people, 31 were observed to be of type AA, 48 of type Aa, and 21 of type aa. Let $L(\theta)$ be the probability of observing this specific sequence of genotypes.

    (a) Explain why
    $$L(\theta) = 2^{48}\theta^{110}(1 - \theta)^{90}$$
    for $0 \leq \theta \leq 1$.

    (b) Find the value of θ which maximizes $L(\theta)$ on $[0, 1]$. What are the corresponding values for the probabilities of AA, Aa, and aa?

21. In the final example of this section, we showed that the surface area of a right circular cylindrical can is minimized when the height of the can is equal to the diameter of the can. Check out a local supermarket to see how many cans satisfy this condition. What other considerations might be important in the design of a can?

22. We have seen that if $x(t)$ is the position of an object moving on a straight line at time t, then the velocity of the object is given by $v(t) = \dot{x}(t)$ and the acceleration is given by $a(t) = \dot{v}(t)$. Hence a is the second derivative of x; that is, $a(t) = \ddot{x}(t)$. Suppose an object is oscillating at the end of a spring so that its position at time t is $x = 3\sin(\pi t)$.

    (a) Find $v(t)$.
    (b) Find $a(t)$.
    (c) Discuss the behavior of the object over the interval $[0, 2]$, taking into account the values of $x(t)$, $v(t)$, and $a(t)$.

Lecture 19: Maximum and Minimum Values

19.1 Maximum and minimum values

Definition   Let D be the domain of the function f. If $f(c) \geq f(x)$ for all x in D, then we say f has an *absolute maximum* at c and we call $f(c)$ the *maximum value* of f on D. If $f(c) \leq f(x)$ for all x in D, then we say f has an *absolute minimum* at c and we call $f(c)$ the *minimum value* of f on D.

Example $f(x) = x^2$ has a minimum value of 0 at $x = 0$, but does not have a maximum value.

Example $f(x) = \sin(x)$ has a a maximum value of 1 at $x = \frac{\pi}{2} + 2\pi n, n = 0, \pm 1, \pm 2, \ldots$ and a minimum value of -1 at $x = \frac{3\pi}{2} + 2\pi n, n = 0, \pm 1, \pm 2, \ldots$.

Example $f(x) = x^2$ on $(0, 1)$ has neither a maximum nor a minimum value.

Example   Let
$$f(x) = \begin{cases} \dfrac{1}{x}, & \text{if } 0 < x \leq 1, \\ 2, & \text{if } x = 0. \end{cases}$$
Then f has a minimum value of 1 at $x = 1$ but does not have a maximum value.

The Extreme Value Theorem   If f is continuous on $[a, b]$, then f has a maximum and a minimum value on $[a, b]$.

19.2 Local maximum and minimum values

Definition   We say f has a *local maximum* at c if $f(c) \geq f(x)$ for all x in some open interval I which contains c. We say f has a *local minimum* at c if $f(c) \leq f(x)$ for all x in some open interval I which contains c.

Suppose f has a local maximum at c and f is differentiable at c. Then, for sufficiently small $h > 0$, we have
$$\frac{f(c+h) - f(c)}{h} \leq 0,$$
which implies that
$$\lim_{h \to 0^+} \frac{f(c+h) - f(c)}{h} \leq 0.$$
Similarly, for sufficiently small $h < 0$,
$$\frac{f(c+h) - f(c)}{h} \geq 0,$$

which implies that

$$\lim_{h \to 0^-} \frac{f(c+h) - f(c)}{h} \geq 0.$$

Thus we must have

$$f'(c) = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h} = 0.$$

Theorem   If f is differentiable at c and f has a local maximum or minimum at c, then $f'(c) = 0$.

Note that if $f(x) = x^3$, then $f'(0) = 0$ even though f does not have either a local minimum or a local maximum at $x = 0$.

Definition   We call a number c a *critical number* of a function f if either f is not differentiable at c or $f'(c) = 0$.

19.3 Finding maximum and minimum values

If f is continuous on $[a, b]$, then the maximum and minimum values of f must occur at either (1) critical numbers in (a, b) or (2) endpoints.

Example   Let $g(t) = \sin(t) + \cos(t)$ on $[0, 2\pi]$. Now

$$g'(t) = \cos(t) - \sin(t),$$

so $g'(t) = 0$ when

$$\cos(t) - \sin(t) = 0,$$

that is, when $\cos(t) = \sin(t)$ (or, equivalently, when $\tan(t) = 1$). Hence $g'(t) = 0$ when $t = \frac{\pi}{4}$ for $t = \frac{5\pi}{4}$. Evaluating at critical numbers and endpoints, we find:
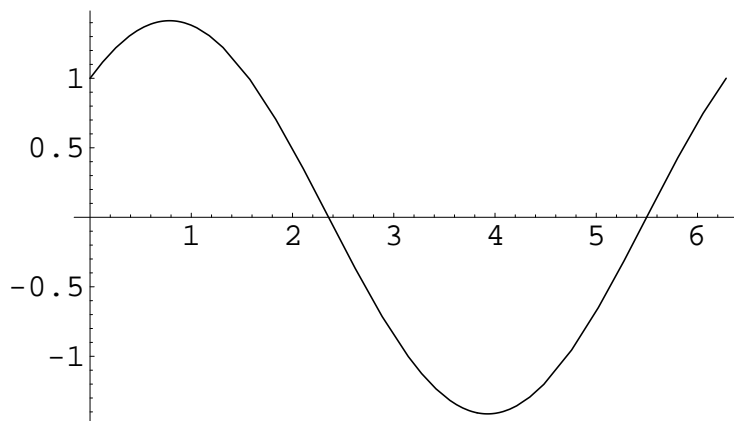
$$g(0) = 1$$

$$g(\frac{\pi}{4}) = \sqrt{2}$$

$$g(\frac{5\pi}{4}) = -\sqrt{2}$$

$$g(2\pi) = 1$$

Thus g has a maximum value of $\sqrt{2}$ at $t = \frac{\pi}{4}$ and a minimum value of $-\sqrt{2}$ at $t = \frac{5\pi}{4}$.
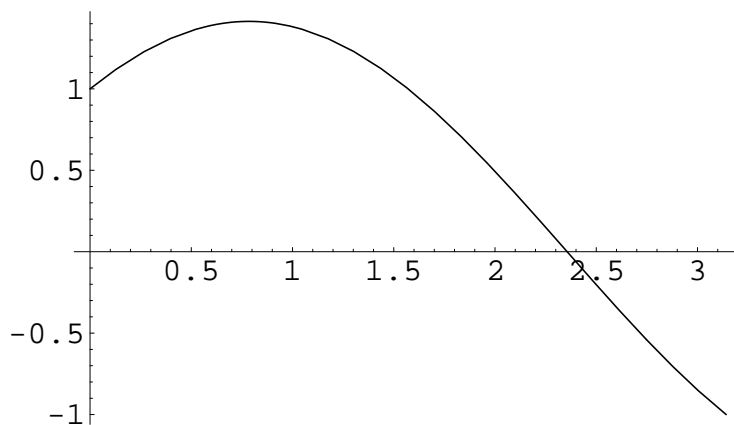
Graph of $g(t) = \sin(t) + \cos(t)$ on $[0, 2\pi]$

Example Let $f(t) = \sin(t) + \cos(t)$ on $[0, \pi]$. Then the only critical number is $\frac{\pi}{4}$. Evaluating at critical numbers and endpoints, we find:

$$f(0) = 1$$

$$f(\frac{\pi}{4}) = \sqrt{2}$$

$$f(\pi) = -1$$

Thus f has a maximum value of $\sqrt{2}$ at $t = \frac{\pi}{4}$ and a minimum value of -1 at $t = \pi$.

Graph of $g(t) = \sin(t) + \cos(t)$ on $[0, \pi]$

Example Let $f(x) = x^{\frac{2}{3}}$ on $[-1, 1]$. Then
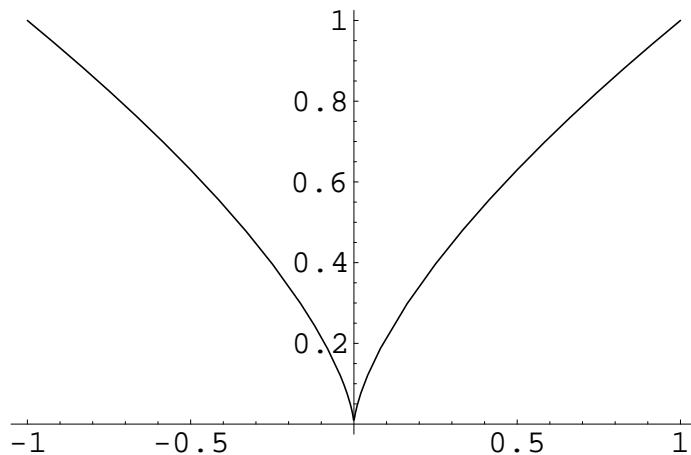
$$f'(x) = \frac{2}{3x^{\frac{1}{3}}},$$

so f is not differentiable at $x = 0$, and this is the only critical number of f. Evaluating at critical numbers and endpoints, we have:

$$f(-1) = 1$$

$$f(0) = 0$$
$$f(1) = 1$$

Thus f has a maximum value of 1 at $x = -1$ and at $x = 1$, and a minimum value of 0 at $x = 0$.

Graph of $f(x) = x^{\frac{2}{3}}$ on $[-1, 1]$

Example Let $f(x) = 3x^4 + 4x^3 - 12x^2 + 2$ on $[-1, 2]$. Then
$$f'(x) = 12x^3 + 12x^2 - 24x = 12x(x^2 + x - 2) = 12x(x + 2)(x - 1).$$

Hence $f'(x) = 0$ when $12x(x+2)(x-1) = 0$, that is, when $x = -2$, $x = 0$, or $x = 1$. Thus the critical numbers in $[-1, 2]$ are 0 and 1. Evaluating at critical numbers and endpoints, we find:
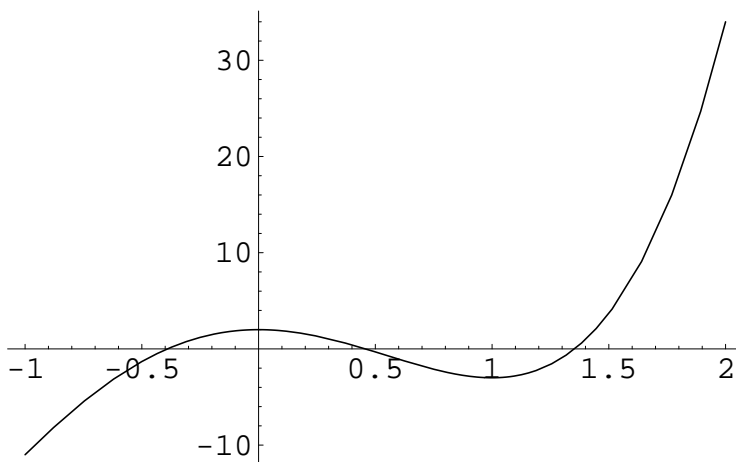
$$f(-1) = -11$$
$$f(0) = 2$$
$$f(1) = -3$$
$$f(2) = 34$$

Thus f has a maximum value of 34 at $x = 2$ and a minimum value of -11 at $x = -1$.

Graph of $f(x) = 3x^4 + 4x^3 - 12x^2 + 2$ on $[-1, 2]$