Difference Equations to **D**ifferential Equations

Section 3.3

Differentiation of Polynomials and Rational Functions

In this section we begin the task of discovering rules for differentiating various classes of functions. By the end of Section 3.5 we will be able to differentiate any algebraic or trigonometric function as a matter of routine without reference to the limits used Section 3.2.

Differentiation of polynomials

We first note that if f is a first degree polynomial, say, $f(x) = ax + b$ for some constants a and b, then f is an affine function and hence its own best affine approximation. Thus $f'(x) = a$ for all x. In particular, if f is a constant function, say, $f(x) = b$ for all x, then $f'(x) = 0$ for all x.

Next we consider the case of a monomial $f(x) = x^n$, where n is a positive integer greater than 1. Then

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{(x+h)^n - x^n}{h}. \qquad (3.3.1)$$

Now

$$(x+h)^n = x^n + nx^{n-1}h + R(h), \qquad (3.3.2)$$

where $R(h)$ represents the remaining terms in the expansion. Since every term in $R(h)$ has a factor of h raised to a power greater than or equal to 2, it follows that $R(h)$ is $o(h)$. Hence we have

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{x^n + nx^{n-1}h + R(h) - x^n}{h} \\ &= \lim_{h \to 0} \frac{nx^{n-1}h + R(h)}{h} \\ &= \lim_{h \to 0} \left(nx^{n-1} + \frac{R(h)}{h} \right) \\ &= nx^{n-1} + \lim_{h \to 0} \frac{R(h)}{h} \\ &= nx^{n-1}. \end{aligned}$$

Since from our previous result $f'(x) = 1$ when $f(x) = x$, this formula also works in the case $n = 1$. Hence we have the following proposition.

Proposition For any positive integer n,

$$\frac{d}{dx} x^n = nx^{n-1}. \qquad (3.3.3)$$

Example If $f(x) = x^3$, then $f'(x) = 3x^2$, as we saw in an example in Section 3.2.

Example Similarly,

$$\frac{d}{dt}t^5 = 5t^4.$$

Hence, for example, the equation of the line tangent to the curve $x = t^5$ at $(-1, -1)$ is

$$x = 5(t+1) - 1,$$

or

$$x = 5t + 4.$$

Once we establish results for the derivative of a constant times a function and for the derivative of the sum of two functions, similar to the results we have for limits, we will be able to easily differentiate any polynomial. So suppose f is a differentiable function and let $k(x) = cf(x)$, where c is any constant. Then

$$\begin{aligned} k'(x) &= \lim_{h \to 0} \frac{k(x+h) - k(x)}{h} \\ &= \lim_{h \to 0} \frac{cf(x+h) - cf(x)}{h} \\ &= \lim_{h \to 0} \frac{c(f(x+h) - f(x))}{h} \\ &= c \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\ &= cf'(x). \end{aligned}$$

That is, the derivative of a constant times a function is the constant times the derivative of the function.

Proposition If f is differentiable and c is any constant, then

$$\frac{d}{dx}(cf(x)) = c\frac{d}{dx}f(x). \qquad (3.3.4)$$

Example If $f(x) = 14x^3$, then

$$f'(x) = (14)(3x^2) = 42x^2.$$

Now suppose f and g are both differentiable functions and let $k(x) = f(x) + g(x)$. Then

$$\begin{aligned} k'(x) &= \lim_{h \to 0} \frac{k(x+h) - k(x)}{h} \\ &= \lim_{h \to 0} \frac{(f(x+h) + g(x+h)) - (f(x) + g(x))}{h} \\ &= \lim_{h \to 0} \left(\frac{f(x+h) - f(x)}{h} + \frac{g(x+h) - g(x)}{h} \right) \\ &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} + \lim_{h \to 0} \frac{g(x+h) - g(x)}{h} \\ &= f'(x) + g'(x). \end{aligned}$$

Hence the derivative of the sum of two functions is the sum of their derivatives. A similar argument would show that the derivative of the difference of two functions is the difference of their derivatives.

Proposition If f and g are both differentiable, then

$$\frac{d}{dx}(f(x)+g(x)) = \frac{d}{dx}f(x) + \frac{d}{dx}g(x) \tag{3.3.5}$$

and

$$\frac{d}{dx}(f(x)-g(x)) = \frac{d}{dx}f(x) - \frac{d}{dx}g(x). \tag{3.3.6}$$

Putting the preceding results together, we are now in a position to easily differentiate any polynomial, as the next examples will illustrate.

Example Suppose $f(x) = 3x^5 - 6x^2 + 2x - 16$. Then

$$\begin{aligned} f'(x) &= \frac{d}{dx}(3x^5 - 6x^2 + 2x - 16) \\ &= \frac{d}{dx}(3x^5) - \frac{d}{dx}(6x^2) + \frac{d}{dx}(2x) - \frac{d}{dx}(16) \\ &= 3\frac{d}{dx}x^5 - 6\frac{d}{dx}x^2 + 2\frac{d}{dx}x - 0 \\ &= (3)(5x^4) - (6)(2x) + (2)(1) \\ &= 15x^4 - 12x + 2. \end{aligned}$$

Example Of course, it is not necessary to write out in detail all the steps in differentiating a polynomial as we did in the preceding example. For example, if $g(t) = 3t^{12} - 6t^2 + t$, then

$$g'(t) = (3)(12t^{11}) - (6)(2t) + 1 = 36t^{11} - 12t + 1.$$

In particular, since $g(1) = -2$ and $g'(1) = 25$, the best affine approximation to g at $t = 1$ is

$$T(t) = 25(t-1) - 2 = 25t - 27.$$

Differentiation of rational functions

We next consider the problem of differentiating the quotient of two functions whose derivatives are already known. In particular, combining this result with our result for polynomials will enable us to easily differentiate any rational function. We might hope that, analogous to the last two results and the related results for limits, the derivative of the quotient of two functions would be equal to the quotient of their derivatives. This turns out not to be true; nevertheless, there is a nice rule for differentiating quotients.

Suppose f and g are both differentiable functions and let $k(x) = \frac{f(x)}{g(x)}$. Then, at all points where $g(x) \neq 0$,

$$\begin{aligned} k'(x) &= \lim_{h\to 0} \frac{k(x+h)-k(x)}{h} \\ &= \lim_{h\to 0} \frac{\dfrac{f(x+h)}{g(x+h)} - \dfrac{f(x)}{g(x)}}{h} \\ &= \lim_{h\to 0} \frac{\dfrac{g(x)f(x+h)-g(x+h)f(x)}{g(x+h)g(x)}}{h} \\ &= \lim_{h\to 0} \frac{g(x)f(x+h)-g(x+h)f(x)}{hg(x)g(x+h)}. \end{aligned}$$

It turns out that by adding and subtracting the term $g(x)f(x)$ (a standard mathematical trick of adding 0) in the numerator, we can simplify this limit into a form that we can evaluate. That is,

$$\begin{aligned} k'(x) &= \lim_{h\to 0} \frac{g(x)f(x+h)-g(x+h)f(x)}{hg(x)g(x+h)} \\ &= \lim_{h\to 0} \frac{g(x)f(x+h)-g(x)f(x)+g(x)f(x)-g(x+h)f(x)}{hg(x)g(x+h)} \\ &= \lim_{h\to 0} \frac{g(x)(f(x+h)-f(x))-f(x)(g(x+h)-g(x))}{hg(x)g(x+h)} \\ &= \lim_{h\to 0} \frac{g(x)\left(\dfrac{f(x+h)-f(x)}{h}\right) - f(x)\left(\dfrac{g(x+h)-g(x)}{h}\right)}{g(x)g(x+h)}. \end{aligned}$$

Now

$$\lim_{h\to 0} g(x)\left(\frac{f(x+h)-f(x)}{h}\right) = g(x)\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = g(x)f'(x), \tag{3.3.7}$$

$$\lim_{h\to 0} f(x)\left(\frac{g(x+h)-g(x)}{h}\right) = f(x)\lim_{h\to 0}\frac{g(x+h)-g(x)}{h} = f(x)g'(x), \tag{3.3.8}$$

and

$$\lim_{h\to 0} g(x)g(x+h) = g(x)\lim_{h\to 0} g(x+h) = g(x)g(x) = (g(x))^2, \tag{3.3.9}$$

where the limits in (3.3.7) and (3.3.8) follow from the differentiability of f and g, while the limit in (3.3.9) follows from the continuity of g (which is a consequence of the differentiability of g). Putting everything together, we have

$$k'(x) = \frac{g(x)f'(x)-f(x)g'(x)}{(g(x))^2}, \tag{3.3.10}$$

a result known as the *quotient rule*.

Quotient Rule If f and g are both differentiable, then

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)\frac{d}{dx}f(x) - f(x)\frac{d}{dx}g(x)}{(g(x))^2} \tag{3.3.11}$$

at all points where $g(x) \neq 0$.

Example Suppose $f(x) = \dfrac{2x+1}{x-2}$. Then

$$\begin{aligned} f'(x) &= \frac{(x-2)\frac{d}{dx}(2x+1) - (2x+1)\frac{d}{dx}(x-2)}{(x-2)^2} \\ &= \frac{(x-2)(2) - (2x+1)(1)}{(x-2)^2} \\ &= \frac{2x-4-2x-1}{(x-2)^2} \\ &= -\frac{5}{(x-2)^2}. \end{aligned}$$

Hence, for example, $f(3) = 7$ and $f'(3) = -5$, so the equation of the line tangent to the graph of f at $(3, 7)$ is

$$y = -5(x-3) + 7,$$

or

$$y = -5x + 22.$$

Example Suppose $g(z) = \dfrac{1}{z^2}$. Then

$$g'(z) = \frac{z^2\frac{d}{dz}(1) - (1)\frac{d}{dz}(z^2)}{z^4} = \frac{(z^2)(0) - 2z}{z^4} = -\frac{2}{z^3}.$$

Note that we may write this result in the form

$$\frac{d}{dz}z^{-2} = -2z^{-3},$$

which is consistent with our previous result

$$\frac{d}{dz}z^n = nz^{n-1}.$$

However, we derived the latter under the assumption that n was a positive integer. We will now show that we can extend this result to the case of negative integer exponents.

Suppose $f(x) = x^n$, where n is a negative integer. Then, using the quotient rule and the fact that $-n > 0$,

$$\begin{aligned} f'(x) &= \frac{d}{dx}x^n \\ &= \frac{d}{dx}\left(\frac{1}{x^{-n}}\right) \\ &= \frac{x^{-n}\frac{d}{dx}(1) - (1)\frac{d}{dx}(x^{-n})}{x^{-2n}} \\ &= \frac{(x^{-n})(0) - (-nx^{-n-1})}{x^{-2n}} \\ &= \frac{nx^{-n-1}}{x^{-2n}} \\ &= nx^{-n-1+2n} \\ &= nx^{n-1}. \end{aligned}$$

We can now state the more general result.

Proposition For any integer $n \neq 0$,

$$\frac{d}{dx}x^n = nx^{n-1}. \tag{3.3.12}$$

Example If $f(x) = \frac{1}{x}$, then

$$f'(x) = \frac{d}{dx}x^{-1} = -x^{-2} = -\frac{1}{x^2}.$$

Example Similarly,

$$\frac{d}{dx}\left(\frac{5}{x^3}\right) = \frac{d}{dx}(5x^{-3}) = -15x^{-4} = -\frac{15}{x^4}.$$

We will eventually see that (3.3.12) holds for rational and irrational exponents as well. We will consider the rational case in Section 3.4, but we will not have the tools for handling the irrational case until we discuss exponential and logarithm functions in Chapter 6.

Differentiation of products

We will close this section with a discussion of a rule for differentiating the product of two functions. Since the product of two rational functions is again a rational function, this will not extend the class of functions that we know how to differentiate routinely. However, this rule will be very useful in the future and, even at the present point, can help simplify some problems.

Suppose f and g are both differentiable and $k(x) = f(x)g(x)$. Then

$$k'(x) = \lim_{h\to 0} \frac{k(x+h) - k(x)}{h} = \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h}. \tag{3.3.13}$$

Adding and subtracting $f(x+h)g(x)$ in the numerator (again, the mathematical trick of adding 0 in a useful manner) will help simplify this limit. Namely,

$$\begin{aligned} k'(x) &= \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x+h)g(x) + f(x+h)g(x) - f(x)g(x)}{h} \\ &= \lim_{h\to 0} \frac{f(x+h)(g(x+h) - g(x)) + g(x)(f(x+h) - f(x))}{h} \\ &= \lim_{h\to 0} \left(f(x+h) \left(\frac{g(x+h) - g(x)}{h} \right) + g(x) \left(\frac{f(x+h) - f(x)}{h} \right) \right). \end{aligned}$$

Now

$$\lim_{h\to 0} f(x+h) \left(\frac{g(x+h) - g(x)}{h} \right) = \lim_{h\to 0} f(x+h) \lim_{h\to 0} \frac{g(x+h) - g(x)}{h} = f(x)g'(x) \tag{3.3.14}$$

and

$$\lim_{h\to 0} g(x) \left(\frac{f(x+h) - f(x)}{h} \right) = g(x) \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = g(x)f'(x), \tag{3.3.15}$$

where, as with the derivation of the quotient rule, we have used the differentiability of f and g as well as the continuity of f in evaluating the limits. Putting everything together, we have

$$k'(x) = f(x)g'(x) + g(x)f'(x). \tag{3.3.16}$$

a result known as the *product rule.*

Product Rule If f and g are both differentiable, then

$$\frac{d}{dx} f(x)g(x) = f(x)\frac{d}{dx}g(x) + g(x)\frac{d}{dx}f(x). \tag{3.3.17}$$

Example If

$$f(x) = (x^4 - 3x^2 + 6x - 3)(6x^3 + 2x + 5),$$

then

$$\begin{aligned} f'(x) &= (x^4 - 3x^2 + 6x - 3)\frac{d}{dx}(6x^3 + 2x + 5) + (6x^3 + 2x + 5)\frac{d}{dx}(x^4 - 3x^2 + 6x - 3) \\ &= (x^4 - 3x^2 + 6x - 3)(18x^2 + 2) + (6x^3 + 2x + 5)(4x^3 - 6x + 6). \end{aligned}$$

Of course, in this example, f is just a polynomial so we could also find f' by multiplying out the two factors of f and differentiating the polynomial term by term as usual. However, the product rule gives us a quicker route to the derivative. Although the result is not simplified into the standard form of a polynomial, for most applications this form is just as useful as any other.

It is worth noting that although we can now differentiate any rational function in theory, in practice our methods may not be the most useful. For example, the function

$$f(x) = (x^2 + 1)^{567}$$

is a polynomial, and so we know how to differentiate it. However, at this point the only way we could perform the differentiation would be to expand $f(x)$ into standard polynomial form and then differentiate term by term. In Section 3.4 we will learn how to handle this problem more directly. At the same time we will extend the class of functions that we can differentiate routinely to include all algebraic functions.

Problems

1. Find the derivative of each of the following functions.

   (a) $f(x) = x^3 + 6x$

   (b) $g(x) = 13x^5 - 6x^2 + 13$

   (c) $g(t) = 3t - 6t^2$

   (d) $y(t) = 4t^3 - 18t + 3$

   (e) $f(t) = (3t - 6)^2$

   (f) $f(x) = (4x + 5)(6x^2 - 1)$

2. Find the derivative of each of the following functions.

   (a) $f(x) = (2x + 1)^2$

   (b) $g(t) = (t^2 - 3)^3$

   (c) $g(x) = \dfrac{x - 3}{2x + 5}$

   (d) $h(s) = \dfrac{2s - s^2}{s^2 + 1}$

   (e) $f(t) = \dfrac{3t^4 - 8t + 1}{2t^3 + 6}$

   (f) $x(t) = \dfrac{3}{t^3} - 16t^2$

   (g) $h(t) = \dfrac{3}{t}$

   (h) $f(x) = \dfrac{41}{3x^7}$

   (i) $h(z) = 8z^3 - \dfrac{1}{2z}$

   (j) $f(s) = \dfrac{31}{s^3} + \dfrac{1}{2s^2} - 16s$

3. For each of the following, make use of the product rule in finding the derivative of the dependent variable with respect to the independent variable.

   (a) $s = (t^2 - 6t + 3)(8t^4 + 6t^2 - 7)$

   (b) $q = (13t^4 + 5t)(3t^5 + 4t^3 + 16t - 31)$

   (c) $y = (x^2 - 2x + 3)(2x^2 + 13x - 6)(3x^2 - 4x + 1)$

   (d) $z = \dfrac{(x^2 - 3x + 6)(8x^2 + 3x - 2)}{x^2 - 6}$

4. Suppose $f(2) = -2$, $f'(2) = 6$, $g(2) = 3$, and $g'(2) = -4$. Find $k'(2)$ for each of the following.

   (a) $k(x) = f(x)g(x)$

   (b) $k(x) = \dfrac{f(x)}{g(x)}$

   (c) $k(x) = f(x)(g(x))^2$

   (d) $k(x) = \dfrac{f(x) - f(x)g(x)}{g(x)}$

5. Suppose an object moves along the x-axis so that its position at time t is $x = -t + \dfrac{t^3}{6}$.

   (a) Find the velocity, $v(t) = \dot{x}(t)$, of the object.

   (b) What is $v(0)$? What does this say about the direction of motion of the object at time $t = 0$?

   (c) When is the object at the origin? What is the velocity of the object when it is at the origin?

   (d) For what values of t is the object moving toward the right?

   (e) For what values of t is the object moving toward the left?

   (f) What is happening at the points where $v(t) = 0$?

   (g) Find the acceleration of the object, $a(t) = \dot{v}(t)$.

   (h) When is the acceleration positive? When is it negative?

   (i) Notice that $v(1) < 0$ and $a(1) > 0$. What does this say about the motion at time $t = 1$?

6. (a) Using only the product rule and the fact that $\dfrac{d}{dx}x = 1$, show that $\dfrac{d}{dx}x^2 = 2x$.

   (b) Now use the product rule to show that $\dfrac{d}{dx}x^3 = 3x^2$.

   (c) Let $n > 1$ and suppose we know that

$$\frac{d}{dx}x^m = mx^{m-1}$$

   for all $m < n$. Use the product rule to show that

$$\frac{d}{dx}x^n = nx^{n-1}.$$