Difference Equations
to
Differential Equations

Section 6.4

Integration of Rational Functions

In this section we will take a more detailed look at the use of partial fraction decompositions in evaluating integrals of rational functions, a technique we first encountered in the inhibited growth model example in the previous section. However, we will not be able to complete the story until after the introduction of the inverse tangent function in Section 6.5.

We begin with a few examples to illustrate how some integration problems involving rational functions may be simplified either by a long division or by a simple substitution.

Example To evaluate $\int \frac{x^2}{x+1}\, dx$, we first perform a long division of $x+1$ into x^2 to obtain

$$\frac{x^2}{x+1} = x - 1 + \frac{1}{x+1}.$$

Then

$$\int \frac{x^2}{x+1}\, dx = \int \left(x - 1 + \frac{1}{x+1}\right) dx = \frac{1}{2}x^2 - x + \log|x+1| + c.$$

Example To evaluate $\int \frac{2x+1}{x^2+x}\, dx$, we make the substitution

$$\begin{aligned} u &= x^2 + x \\ du &= (2x+1)dx. \end{aligned}$$

Then

$$\int \frac{2x+1}{x^2+x}\, dx = \int \frac{1}{u}\, du = \log|u| + c = \log|x^2+x| + c.$$

Example To evaluate $\int \frac{x}{x+1}\, dx$, we perform a long division of $x+1$ into x to obtain

$$\frac{x}{x+1} = 1 - \frac{1}{x+1}.$$

Then

$$\int \frac{x}{x+1}\, dx = \int \left(1 - \frac{1}{x+1}\right) dx = x - \log|x+1| + c.$$

Alternatively, we could evaluate this integral with the substitution

$$\begin{aligned} u &= x + 1 \\ du &= dx. \end{aligned}$$

With this substitution, $x = u - 1$, so we have

$$\begin{aligned} \int \frac{x}{x+1}\, dx &= \int \frac{u-1}{u}\, du \\ &= \int \left(1 - \frac{1}{u}\right) du \\ &= u - \log|u| + c \\ &= x + 1 - \log|x+1| + c. \end{aligned}$$

Note that this is the same answer we obtained above, although with a different constant of integration.

Partial fraction decomposition: Distinct linear factors

Now we consider the general problem of evaluating

$$\int \frac{f(x)}{g(x)}\, dx$$

where both f and g are polynomials. We will assume that the degree of g is less than the degree of f. As illustrated in the first and third examples above, if this is not the case, we can first perform a long division to simplify the quotient into the form of a polynomial plus a remainder term which is a rational function with numerator of degree less than the denominator. To begin we will suppose that g factors completely into n distinct linear factors. That is, suppose there are constants $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ such that

$$g(x) = (a_1x + b_1)(a_2x + b_2)\cdots(a_nx + b_n), \tag{6.4.1}$$

where the factors on the right are all distinct. From a theorem of linear algebra, which we will not attempt to prove here, there exist constants $A_1, A_2, \ldots, A_n$ such that

$$\frac{f(x)}{g(x)} = \frac{A_1}{a_1x + b_1} + \frac{A_2}{a_2x + b_2} + \cdots + \frac{A_n}{a_nx + b_n}. \tag{6.4.2}$$

The expression on the right of (6.4.2) is called the *partial fraction decomposition* of $\dfrac{f(x)}{g(x)}$. Once the constants $A_1, A_2, \ldots, A_n$ are determined, the evaluation of

$$\int \frac{f(x)}{g(x)}\, dx$$

becomes a routine problem. The next examples will illustrate one method for finding these constants.

Example To evaluate $\int \frac{1}{(x-2)(x-3)}\,dx$, we need to find constants A and B such that

$$\frac{1}{(x-2)(x-3)} = \frac{A}{x-2} + \frac{B}{x-3}.$$

Combining the terms on the right, we have

$$\frac{1}{(x-2)(x-3)} = \frac{A(x-3)+B(x-2)}{(x-2)(x-3)}.$$

Now two rational functions with equal denominators are equal only if their numerators are also equal; hence we must have

$$1 = A(x-3) + B(x-2)$$

for all values of x. In particular, for $x = 2$ we obtain

$$1 = -A,$$

from which it follows that $A = -1$, and for $x = 3$ we have

$$1 = B.$$

Thus

$$\frac{1}{(x-2)(x-3)} = -\frac{1}{x-2} + \frac{1}{x-3},$$

so

$$\begin{aligned}\int \frac{1}{(x-2)(x-3)}\,dx &= -\int \frac{1}{x-2}\,dx + \int \frac{1}{x-3}\,dx \\ &= -\log|x-2| + \log|x-3| + c.\end{aligned}$$

Example To evaluate $\int \frac{3x}{(x+5)(2x-1)}\,dx$, we need to find constants A and B such that

$$\frac{3x}{(x+5)(2x-1)} = \frac{A}{x+5} + \frac{B}{2x-1}.$$

Combining the terms on the right, we have

$$\frac{3x}{(x+5)(2x-1)} = \frac{A(2x-1)+B(x+5)}{(x+5)(2x-1)}.$$

As before, it follows that

$$3x = A(2x-1) + B(x+5)$$

for all values of x. In particular, for $x = -5$ we obtain

$$-15 = -11A,$$

from which it follows that

$$A = \frac{15}{11},$$

and for $x = \frac{1}{2}$ we have

$$\frac{3}{2} = \frac{11}{2}B,$$

from which it follows that

$$B = \frac{3}{11}.$$

Hence

$$\frac{3x}{(x+5)(2x-1)} = \frac{15}{11}\frac{1}{x+5} + \frac{3}{11}\frac{1}{2x-1},$$

so

$$\begin{aligned}\int \frac{1}{(x+5)(2x-1)}\,dx &= \frac{15}{11}\int \frac{1}{x+5}\,dx + \frac{3}{11}\frac{1}{2x-1}\,dx \\ &= \frac{15}{11}\log|x+5| + \frac{3}{22}\log|2x-1| + c.\end{aligned}$$

Partial fraction decomposition: Repeated linear factors

Returning to the general problem of evaluating

$$\int \frac{f(x)}{g(x)}\,dx,$$

where f and g are both polynomials and the degree of f is less than the degree of g, we will now consider the case where g factors completely into linear factors, allowing for the possibility that one or more of these factors may be repeated. Specifically, suppose the factor $ax+b$ occurs n times in the factorization of g. Then the partial fraction decomposition of $\frac{f(x)}{g(x)}$ must contain a sum of terms of the form

$$\frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \cdots + \frac{A_n}{(ax+b)^n}, \tag{6.4.3}$$

for some constants $A_1, A_2, \ldots, A_n$, in addition to similar terms for every other factor of g. This is best illustrated in an example.

Example To evaluate $\frac{x+1}{(x-1)^3(x-2)}\,dx$, we need to find constants A, B, C, and D such that

$$\frac{x+1}{(x-1)^3(x-2)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{(x-1)^3} + \frac{D}{x-2}. \tag{6.4.4}$$

That is, this partial fraction decomposition contains three terms corresponding to the factor $x-1$, since it is repeated three times, and only one term corresponding to the factor $x-2$, since it occurs only once. Moreover, the degrees of the denominators of the terms for $x-1$ increase from 1 to 3. Now combining the terms on the right of (6.4.4), we have

$$\frac{x+1}{(x-1)^3(x-2)} = \frac{A(x-1)^2(x-2)+B(x-1)(x-2)+C(x-2)+D(x-1)^3}{(x-1)^3(x-2)}.$$

Again, it follows that

$$x+1 = A(x-1)^2(x-2)+B(x-1)(x-2)+C(x-2)+D(x-1)^3 \tag{6.4.5}$$

for all values of x. However, because of the repeated factors, we cannot choose values for x which will isolate each of the constants one at a time as we did in the previous examples. Instead, we will illustrate another technique for finding the constants. By multiplying out (6.4.5) and collecting terms, we obtain

$$\begin{aligned} x+1 &= A(x^3-4x^2+5x-2)+B(x^2-3x+2)+C(x-2)+D(x^3-3x^2+3x-1) \\ &= (A+D)x^3+(-4A+B-3D)x^2+(5A-3B+C+3D)x-2A+2B-2C-D \end{aligned}$$

for all values of x. Since two polynomials are equal only if they have equal coefficients, we can equate the coefficients of $x+1$ with the coefficients of the polynomial on the right to obtain the four equations

$$\begin{aligned} A+D&=0 \\ -4A+B-3D&=0 \\ 5A-3B+C+3D&=1 \\ -2A+2B-2C-D&=1. \end{aligned} \tag{6.4.6}$$

From the first equation we learn that

$$D=-A.$$

Substituting this into the second equation gives us

$$B=A.$$

Substituting both of these values into the third equation results in

$$C=A+1.$$

Finally, substituting for D, B, and C in the fourth equation gives us

$$-2A+2A-2(A+1)+A=1,$$

which gives us $A = -3$. Hence $B = -3$, $C = -2$, and $D = 3$. Thus

$$\begin{aligned}\int \frac{x+1}{(x-1)^3(x-2)}\,dx &= -\int \frac{3}{(x-1)}\,dx - \int \frac{3}{(x-1)^2}\,dx \\ &\quad - \int \frac{2}{(x-1)^3}\,dx + \int \frac{3}{x-2}\,dx \\ &= -3\log|x-1| + \frac{3}{x-1} + \frac{1}{(x-1)^2} + 3\log|x-2| + c.\end{aligned}$$

Note that in solving for A, B, C, and D, we could have first substituted $x = 1$ and $x = 2$ into (6.4.5) to obtain values for C and D, respectively. These values could have then been used to simplify (6.4.6) before solving for A and B.

The Fundamental Theorem of Algebra states that every polynomial factors into a product of linear factors and irreducible quadratic factors; hence, to complete the story of integrating rational functions, we need to consider the case where the factorization of the denominator includes irreducible quadratic factors. However, we will learn in Section 6.5 that for an irreducible quadratic polynomial g,

$$\int \frac{1}{g(x)}\,dx$$

involves the inverse tangent function. Thus we need to discuss the inverse trigonometric functions before continuing the story of integrating rational functions.

Problems

1. Evaluate each of the following integrals.

   (a) $\displaystyle\int \frac{x-1}{x}\,dx$ (b) $\displaystyle\int \frac{x}{x-1}\,dx$

   (c) $\displaystyle\int \frac{3x^2}{x-2}\,dx$ (d) $\displaystyle\int \frac{x^3+1}{x+2}\,dx$

   (e) $\displaystyle\int \frac{4x+1}{2x^2+x-3}\,dx$ (f) $\displaystyle\int \frac{x+2}{x^2+4x+1}\,dx$

2. Evaluate each of the following integrals.

   (a) $\displaystyle\int \frac{1}{(x+2)(x-4)}\,dx$ (b) $\displaystyle\int \frac{3}{(x-3)(x-7)}\,dx$

   (c) $\displaystyle\int \frac{3x}{(2x+3)(x+1)}\,dx$ (d) $\displaystyle\int \frac{3x+1}{(x-2)(x+3)}\,dx$

   (e) $\displaystyle\int \frac{x}{x^2+x-6}\,dx$ (f) $\displaystyle\int \frac{3x}{(x+2)(x-3)(x+1)}\,dx$

   (g) $\displaystyle\int \frac{3}{x^2+5x+6}\,dx$ (h) $\displaystyle\int \frac{3x+2}{(x^2-4)(x^2-9)}\,dx$

3. Evaluate each of the following integrals.

   (a) $\displaystyle\int \frac{1}{(x-1)^2}\,dx$

   (b) $\displaystyle\frac{1}{(x-1)^2(x+2)}\,dx$

   (c) $\displaystyle\int \frac{x}{x^2+2x+1}\,dx$

   (d) $\displaystyle\int \frac{3x+1}{(x+2)^3(x-1)}\,dx$

   (e) $\displaystyle\int \frac{5}{(x+2)^3}\,dx$

   (f) $\displaystyle\int \frac{4}{(x^2-4)^2}\,dx$

   (g) $\displaystyle\int \frac{3x^2}{(x+1)^2(x-3)}\,dx$

   (h) $\displaystyle\int \frac{5x-1}{(2x+1)^2(x+2)}\,dx$

4. Evaluate each of the following integrals.

   (a) $\displaystyle\int \frac{1}{(3x+2)^2}\,dx$

   (b) $\displaystyle\int \frac{3}{x^2+7x+10}\,dx$

   (c) $\displaystyle\int \frac{9x^2-4x}{3x^3-2x^2+5}\,dx$

   (d) $\displaystyle\int \frac{2x}{(x^2-1)(x^2-4)}\,dx$

   (e) $\displaystyle\int_{-1}^{1} \frac{1}{x^2-4}\,dx$

   (f) $\displaystyle\int_{0}^{1} \frac{1}{x^2-x-6}\,dx$

   (g) $\displaystyle\int \frac{4x+5}{(x-2)^2(x+5)}\,dx$

   (h) $\displaystyle\int \frac{x^3}{x^2-1}\,dx$

5. Solve the differential equation

$$\dot{x}(t) = (x(t)-1)(x(t)+1)$$

   using the method used to solve the logistic differential equation in Section 6.3. Assume $x(0) = 0$ and $-1 < x(t) < 1$ for all t.