Difference Equations to **D**ifferential Equations

Section 5.6

Infinite Series: Absolute Convergence

At this point we have limited our study of series primarily to those series having nonnegative terms, the only exceptions being some geometric series and series which are multiples of series with nonnegative terms. In this section we shall consider the more general question of series with negative as well as positive terms.

An important consideration when looking at the behavior of an arbitrary series

$$\sum_{n=1}^{\infty} a_n \tag{5.6.1}$$

is the behavior of the related series

$$\sum_{n=1}^{\infty} |a_n|. \tag{5.6.2}$$

Of course, if all the terms of (5.6.1) are nonnegative, then (5.6.1) and (5.6.2) are the same series. In any case, (5.6.2) has all nonnegative terms, so we may use our results of the last three sections to help determine whether or not it converges. Suppose that, by one method or another, we have shown that (5.6.2) converges. Then, since

$$0 \le a_n + |a_n| \le 2|a_n| \tag{5.6.3}$$

for any n, we know, by the comparison test, that the series

$$\sum_{n=1}^{\infty} (a_n + |a_n|) \tag{5.6.4}$$

converges. Hence

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} (a_n + |a_n|) - \sum_{n=1}^{\infty} |a_n| \tag{5.6.5}$$

converges since it is the difference of two convergent series. That is, the convergence of (5.6.2) implies the convergence of (5.6.1).

Proposition If $\sum_{n=1}^{\infty} |a_n|$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

Definition The series $\sum_{n=1}^{\infty} a_n$ is said to *converge absolutely* if the series $\sum_{n=1}^{\infty} |a_n|$ converges.

With this terminology, the previous proposition says that any series which converges absolutely also converges. We shall see later that the converse of this statement does not hold; namely, there are series which converge, but do not converge absolutely.

Example The series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} = -1 + \frac{1}{4} - \frac{1}{9} + \frac{1}{16} - \frac{1}{25} + \cdots$$

converges absolutely since the series

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^n}{n^2} \right| = \sum_{n=1}^{\infty} \frac{1}{n^2}$$

converges. In particular, it follows that

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$$

converges.

Example The series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots,$$

known as the *alternating harmonic series*, is not absolutely convergent since

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^{n+1}}{n} \right| = \sum_{n=1}^{\infty} \frac{1}{n}$$

is the harmonic series, which diverges. Hence the previous proposition does not provide any information on the behavior of the alternating harmonic series itself. We shall see below that in fact the alternating harmonic series converges even though it is not absolutely convergent.

In general, determining whether a series which is not absolutely convergent is convergent or divergent is a difficult problem. However, there is one particular type of series for which we have, under certain conditions, a simple test. These series are the *alternating series*, the series which, like those in the previous examples, alternate in sign from one term to the next.

Definition A series in which the terms are alternately positive and negative is called an *alternating series.*

Now suppose $\sum_{n=1}^{\infty} a_n$ is an alternating series which satisfies the following two conditions:

(1) $|a_{n+1}| \leq |a_n|$ for $n = 1, 2, 3, \ldots,$

(2) $\lim\limits_{n \to \infty} |a_n| = 0.$

For the sake of the discussion we will assume that $a_1 > 0$, although that will not affect our conclusion. If s_n is the nth partial sum of this series, then

$$s_1 = a_1, \tag{5.6.6}$$

and, since $a_2 < 0$,

$$s_2 = a_1 + a_2 = s_1 + a_2 < s_1. \tag{5.6.7}$$

Next, since $a_3 > 0$,

$$s_3 = a_1 + a_2 + a_3 = s_2 + a_3 > s_2. \tag{5.6.8}$$

Moreover, condition (1) implies $a_2 + a_3 \leq 0$, from which it follows that

$$s_3 = a_1 + a_2 + a_3 = s_1 + a_2 + a_3 \leq s_1. \tag{5.6.9}$$

Thus we have $s_2 \leq s_3 \leq s_1$. Next,

$$s_4 = s_3 + a_4 < s_3 \tag{5.6.10}$$

since $a_4 < 0$ and

$$s_4 = s_2 + a_3 + s_4 \geq s_2 \tag{5.6.11}$$

since $a_3 + a_4 \geq 0$. Thus $s_2 \leq s_4 \leq s_3 \leq s_1$. For the next step,

$$s_5 = s_4 + a_5 > s_4 \tag{5.6.12}$$

since $a_5 > 0$ and

$$s_5 = s_3 + a_4 + a_5 \leq s_3 \tag{5.6.13}$$

since $a_4 + a_5 \leq 0$. Thus $s_2 \leq s_4 \leq s_5 \leq s_3 \leq s_1$. Continuing in this way, we see that

$$s_2 \leq s_4 \leq s_6 \leq s_5 \leq s_3 \leq s_1 \tag{5.6.14}$$

and

$$s_2 \leq s_4 \leq s_6 \leq s_7 \leq s_5 \leq s_3 \leq s_1. \tag{5.6.15}$$

In general, for any positive integer n,

$$s_2 \leq s_4 \leq \cdots \leq s_{2n} \leq \cdots \leq s_{2n-1} \leq \cdots \leq s_5 \leq s_3 \leq s_1. \tag{5.6.16}$$

That is, for $n = 1, 2, 3, \ldots$, $\{s_{2n}\}$ is a bounded increasing sequence and $\{s_{2n-1}\}$ is a bounded decreasing sequence. Thus both sequences have limits, say

$$\lim_{n \to \infty} s_{2n} = L \tag{5.6.17}$$

and

$$\lim_{n \to \infty} s_{2n-1} = M. \tag{5.6.18}$$

But then

$$L - M = \lim_{n\to\infty} s_{2n} - \lim_{n\to\infty} s_{2n-1} = \lim_{n\to\infty}(s_{2n} - s_{2n-1}) = \lim_{n\to\infty} a_{2n} = 0,$$

where the final equality follows from condition (2). Hence $L = M$, so

$$\lim_{n\to\infty} s_n = L. \tag{5.6.19}$$

In other words, $\sum_{n=1}^{\infty} a_n$ converges. This conclusion, known as *Leibniz's theorem*, gives a simple criterion for determining the convergence of some alternating series.

Leibniz's theorem Suppose $\sum_{n=1}^{\infty} a_n$ is an alternating series for which $|a_{n+1}| \leq |a_n|$ for $n = 1, 2, 3, \ldots$. If

$$\lim_{n\to\infty} |a_n| = 0, \tag{5.6.20}$$

then $\sum_{n=1}^{\infty} a_n$ converges.

Example The alternating harmonic series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$$

satisfies the conditions of Leibniz's theorem: If we let

$$a_n = \frac{(-1)^{n+1}}{n},$$

$n = 1, 2, 3, \ldots$, then

$$|a_{n+1}| = \frac{1}{n+1} < \frac{1}{n} = |a_n|$$

and

$$\lim_{n\to\infty} |a_n| = \lim_{n\to\infty} \frac{1}{n} = 0.$$

Thus, as we claimed earlier, the alternating harmonic series converges.

Definition A series which converges but does not converge absolutely is said to *converge conditionally.*

The previous example shows that the alternating harmonic series is an example of a series which converges conditionally.

From the discussion prior to Leibniz's theorem, we see that if $\sum_{n=1}^{\infty} a_n$ satisfies the conditions of Leibniz's theorem, $a_1 > 0$, s_n is its nth partial sum, and

$$s = \sum_{n=1}^{\infty} a_n, \tag{5.6.21}$$

then we must have

$$s_2 \le s_4 \le \cdots \le s_6 \le \cdots \le s \le \cdots \le s_5 \le s_3 \le s_1. \tag{5.6.22}$$

Note that for any positive integer n we have $s_{n+1} \le s \le s_n$ if n is odd and $s_n \le s \le s_{n+1}$ if n is even. Thus, in either case,

$$|s - s_n| \le |s_{n+1} - s_n| = |a_{n+1}|, \tag{5.6.23}$$

a result which also holds if $a_1 < 0$

Proposition Suppose $\sum_{n=1}^{\infty} a_n$ is a convergent alternating series for which $|a_{n+1}| \le |a_n|$ for $n = 1, 2, 3, \ldots$. If

$$s = \sum_{n=1}^{\infty} a_n \tag{5.6.24}$$

and

$$s_n = \sum_{j=1}^{n} a_j, \tag{5.6.25}$$

then, for any $n = 1, 2, 3, \ldots$,

$$|s - s_n| \le |a_{n+1}|. \tag{5.6.26}$$

Hence for those alternating series which satisfy the conditions of the proposition, the error committed in approximating the sum of the series by a particular partial sum is no greater in absolute value than the absolute value of the next term in the series.

Example For the alternating harmonic series, if

$$s = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$$

and

$$s_n = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{(-1)^{n+1}}{n} = \sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{j},$$

then

$$|s - s_n| \le \frac{1}{n+1}$$

for $n = 1, 2, 3, \ldots$. For example

$$s_{100} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{1}{99} - \frac{1}{100} = 0.688172,$$

so

$$|s - s_{100}| \le \frac{1}{101} = 0.009901,$$

where both results have been rounded to 6 decimal places. In other words, the sum of the alternating harmonic series differs from 0.688172 by less than 0.009901. In fact, since the next term in the series is positive, we know that s must lie between 0.688172 and

$$0.688172 + 0.009901 = 0.698073.$$

We will see in Section 6.2 that the sum of the alternating harmonic series is exactly the natural logarithm of 2, which, to 6 decimal places, is 0.693147

Problems

1. For each of the following infinite series, answer the questions: Does the series converge absolutely? Does the series converge conditionally? Does the series converge?

   (a) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^3}$ (b) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$

   (c) $\displaystyle\sum_{n=1}^{\infty} \frac{3n^2-1}{4n^2+2}$ (d) $\displaystyle\sum_{n=0}^{\infty} \frac{(-3)^n}{n!}$

   (e) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n+3}}$ (f) $\displaystyle\sum_{n=3}^{\infty} (-1)^n \pi^n$

   (g) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n n!}{2^n}$ (h) $\displaystyle\sum_{n=1}^{\infty} \left(-\frac{2}{3}\right)^{n-1}$

2. For each of the following infinite series, answer the questions: Does the series converge absolutely? Does the series converge conditionally? Does the series converge?

   (a) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(n^2+1)}{3n^5-2}$ (b) $\displaystyle\sum_{n=0}^{\infty} -\frac{3}{5^n}$

   (c) $\displaystyle\sum_{n=0}^{\infty} \frac{3^{2n}}{(2n)!}$ (d) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n+1}}{(2n+1)!}$

   (e) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n}}{(2n)!}$ (f) $\displaystyle\sum_{n=14}^{\infty} \frac{(-2)^n}{\sqrt{n+1}}$

   (g) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n(n+1)}{2n-1}$ (h) $\displaystyle\sum_{n=2}^{\infty} \frac{1-n}{2n^2}$

3. (a) Approximate

$$s = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!}$$

   using

$$s_{15} = \sum_{n=0}^{15} \frac{(-1)^n}{n!}.$$

(b) Find an upper bound for the error in approximating s by s_{15}.

(c) Find the smallest n such that the absolute value of the error in approximating s by

$$s_n = \sum_{j=0}^{n} \frac{(-1)^j}{j!}$$

is less than 0.000001. What is this approximation?

4. (a) Approximate

$$s = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n2^n}$$

by

$$s_{50} = \sum_{n=1}^{50} \frac{(-1)^{n+1}}{n2^n}.$$

(b) Find an upper bound for the absolute value of the error in approximating s by s_{50}.

(c) Find the smallest n such that the absolute value of the error in approximating s by

$$s_n = \sum_{j=1}^{n} \frac{(-1)^{j+1}}{j2^j}$$

is less than 0.0001. What is this approximation?

5. In our development of Leibniz's theorem, we assumed that $a_1 > 0$. Discuss the changes which must be made in the discussion if $a_1 < 0$.