Proof by Cases

When writing a proof by cases, be sure to do the following:

- Verify that your set of cases is complete — that it accounts for all possibilities.
- Clearly identify your cases.
- Address each case separately. Do not address them in parallel.
- Have at least one concluding sentence after you have addressed all cases.

1 Example 1

If n is an integer, then $n^3 + n$ is even.

1.1 Proof in Symbols

Case 1: If n is even, then $n^3 + n$ is even.

Given	Inferred	Rule of Inference
n is even	$\exists k \in \mathbb{Z} \mid 2k = n$.	*definition of even*
	$n = 2k$ for some integer k	*existential instantiation*
	$n^3 = 8k^3$	*algebra — cube $n = 2k$*
	$n^3 + n = 8k^3 + 2k$	*algebra — substitute*
	$n^3 + n = 2(4k^2 + k)$	*algebra — factor out a 2*
	Let $l = (4k^2 + k)$	*closure of integers*
	$n^3 + n = 2l$	*algebra — substitution*
	$n^3 + n$ is even	*definition of even*

Case 2: If n is odd, then $n^3 + n$ is even.

Given	Inferred	Rule of Inference
n is odd	$\exists k \in \mathbb{Z} \mid 2k + 1 = n$.	*definition of odd*
	$n = 2k + 1$ for some integer k	*existential instantiation*
	$n^3 = 8k^3 + 12k^2 + 6k + 1$	*algebra — cube $n = 2k + 1$*
	$n^3 + n = (8k^3 + 12k^2 + 6k + 1) + (2k + 1)$	*algebra — substitution*
	$n^3 + n = (8k^3 + 12k^2 + 8k + 2)$	*algebra — collect like terms*
	$n^3 + n = 2(4k^3 + 6k^2 + 4k + 1)$	*algebra — factor out a 2*
	Let $l = (4k^3 + 6k^2 + 4k + 1)$	*closure of integers*
	$n^3 + n = 2l$	*algebra — substitution*
	$n^3 + n$ is even	*definition of even*

Proof: if n is an integer, then $n^3 + 3$ is even.

	Given	Previous knowledge	
1	$n \in \mathbb{Z}$	All integers are either even or odd.	*basic arithmetic*
2		if n is even, then $n^3 + n$ is even	*Lemma 1*
3		if n is odd, then $n^3 + n$ is even	*Lemma 2*

	Inferred	Rule of inference
4	n is either even or odd	*Universal instantiation of 1 (p174)*
5	(n even $\rightarrow n^3 + n$ even) $\wedge$ (n odd $\rightarrow n^3 + n$ odd)	*conjunction of 2 and 3.*
6	(n even $\vee$ n odd) $\rightarrow n^3 + n$ even	*logical equivalence with 5*
7	$n^3 + n$ even	*modes ponens of 4 and 6*

1.2 Proof in Words

Let n be an integer. We will prove that $n^3 + 3$ is even by considering two cases:

- n is even.
- n is odd.

First consider the case where n is even. From the definition of even, we know that $n = 2k$ for some integer k. We cube $n = 2k$ to get $n^3 = 8k^3$; then we add $n = 2k + 1$ to get $n^3 + n = 8k^3 + 2k$. Next, we factor out a 2 and get $n^3 + n = 2(4k^3 + k)$. Let $l = 4k^3 + k$. We can now see that $n^3 + n = 2l$ for some integer l. Thus, from the definition of even, we know that $n^3 + n$ is even.

Next consider the case where n is odd. From the definition of odd, we know that $n = 2k + 1$ for some integer k. We cube $n = 2k + 1$ to get $n^3 = 8k^3 + 12k^2 + 6k + 1$; then, we add $n = 2k + 1$ to get that $n^3 + n = 8k^3 + 12k^2 + 8k + 2$. We then factor out a 2 to see that $n^3 + n = 2(4k^3 + 6k^2 + 4k + 1)$. Let $l = (4k^3 + 6k^2 + 4k + 1)$. We now have an integer l such that $2l = n^3 + n$. Thus, from the definition of even, we know that $n^3 + n$ is even.

Because we have shown that $n^3 + n$ is even when n is either even or odd, we have shown that $n^3 + n$ is even for all integers $\blacksquare$