Indirect Proof – Contrapositive

1 Example 1

Definition: An integer n is *odd* if and only if there exists an integer k such that $n = 2k + 1$.

If n^3 is even, then n is even.

If we try to prove this directly, we don't get very far. We infer that $n^3 = 2k$ for some integer k; but, we don't know what to do next.

Instead we can use the fact that $p \to q \Leftrightarrow \neg q \to \neg p$. This means that if we prove that *if n is odd, then n^3 is odd* is true, then we have proven that *if n^3 is even, then n is even* is also true.

1.1 Proof in Symbols

Given	Inferred	Rule of Inference
n is odd.	$\exists k \in \mathbb{Z} \mid 2k + 1 = n$	*definition of odd.*
	$n = 2k + 1$ for some integer k	*existential instantiation.*
	$n^3 = (2k + 1)^3$	*algebra — cube $n = 2k + 1$*
	$n^3 = 8k^3 + 12k^2 + 6k + 1$	*algebra — expand*
	$n^3 = 2(4k^3 + 6k^2 + 3k) + 1$	*algebra — factor out 2*
	Let l be an integer such that $l = 4k^3 + 6k^2 + 3k$	*closure of integers*
	$n^3 = 2l + 1$	*algebra — substitution*
	n^3 is odd	*definition of odd*

1.2 Proof in Words

We will prove that if n^3 is even then n is even indirectly using the contrapositive. That is, we will prove that if n is odd, then n^3 is odd.

Assume that n is odd. From the definition of odd, we know that there exists an integer k such that

$$n = 2k + 1 \tag{1}$$

We then cube equation 1 to see that

$$n^3 = 8k^3 + 8k^2 + 6k + 1 \tag{2}$$

Next, we factor out a 2 to get

$$n^3 = 2(4k^3 + 3k^2 + k) + 1 \tag{3}$$

If we let $l = 4k^3 + 3k^2 + k$ and substitute l into equation 3, we see that $n^3 = 2l + 1$. Thus, by definition of odd, n^3 is odd as desired. ■