1 One - To - One

We say a function $f : A \to B$ is *one-to-one*, or *injective*, if f always maps different elements of A to different elements of B. Formally, $f : A \to B$ is injective if and only if $\forall x, y \in A, [(x \neq y) \to f(x) \neq f(y)]$. Intuitively, this means that if we give f different inputs, we get different outputs. Consider these examples:

- $f : States \to Cities$ defined by $f(x) =$ capital of x: Because each U.S. city is the capital of at most one state, f is one-to-one (i.e. maps exactly one city to each state).
- $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x + 1$ is also one-to-one.
- $f : Cities \to States$ defined by $f(x) =$ location of x is not one-to-one because $f(Atlanta) = f(Macon) = Georgia$.
- $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$ is not one-to-one because $f(-1) = f(1) = 1$.

If we were to draw lines to connect elements of A and B, each element of A would be connected to *exactly* one element of B (this is the definition of function), and each element of B would be connected to *at most* one element of A. Note that it is acceptable for some elements of B to not be connected to any elements of A. Refer to figures 3, 4, and 5 on pages 59-61 of the textbook.

1-a Proving f is Injective

To prove that $f : A \to B$ is injective we must prove that, for every pair $x, y \in A$, if $x \neq y$, then $f(x) \neq f(y)$.

The general method of proving statements with universal quantifiers (i.e. "$\forall x, P(x)$"), is to prove $P(x)$ true for an arbitrary x. Universal generalization then allows us to conclude that $P(x)$ is true for all x. Specifically, to prove that f is injective we will prove that for arbitrary elements $x, y \in A$, if $x \neq y$, then $f(x) \neq f(y)$.[1]

When dealing with specific functions (i.e. functions with a definition, such as $f(x) = x^2$, as opposed to a general function $f : A \to B$), it is often difficult to apply rules of inference to the statement $x \neq y$. However, recall that an implication and its contrapositive are logically equivalent. This means that in the case of injective functions, tthe following propositions have the same truth value:

- $x \neq y \to f(x) \neq f(y)$
- $\neg(f(x) \neq f(y)) \to \neg(x \neq y)$
- $f(x) = f(y) \to x = y$

Therefore, instead of proving that if $x \neq y$, then $f(x) \neq f(y)$, we may instead prove that if $f(x) = f(y)$, then $x = y$.

[1] Note that x and y are *arbitrary* elements of A, not *any* elements of A. We do not assign x and y specific values; to the contrary, the only information we assume about x and y is that they are elements of A.

1-a-i Proof Outline

In summary, to prove $f : A \to B$ is injective:

1. Let x, and y be arbitrary (and not necessarily unique) elements of A.[2]
2. Prove that either
   - if $x \neq y$ then $f(x) \neq f(y)$; or
   - if $f(x) = f(y)$ then $x = y$.
3. State that because the proposition in step 2 is true for arbitrary x and y, it is true for all x and y.
4. Conclude that f is injective.

1-a-ii Example Proof

We will prove that $f : \mathbb{R}^+ \to \mathbb{R}^+$ defined by $f(x) = x^2$ is one to one.

Let x and y be arbitrary elements of $\mathbb{R}^+$. We will use the contrapositive to show that if $x \neq y$, then $f(x) \neq f(y)$. In other words, we will prove that if $f(x) = f(y)$, then $x = y$.

Assume that $f(x) = f(y)$. The definition of f tells us that $x^2 = y^2$. This is equivalent to saying $x^2 - y^2 = 0$. We can factor $x^2 - y^2$ into $(x - y)(x + y)$ and see that $(x - y)(x + y) = 0$. We know that the product of two real numbers ab is zero only if at least one of a or b is zero. Therefore, either $(x - y) = 0$, or $(x + y) = 0$. This means that either $x = y$ or $x = -y$. Because x and y are both positive by definition,[3] the second case is impossible. Therefore, we know that $x = y$ as desired.

We have thus shown that for arbitrary values of x and y, if $x \neq y$, then $f(x) \neq f(y)$. We may therefore conclude that for any pair $x, y \in A$, if $x \neq y$, then $f(x) \neq f(y)$. Hence, f is one-to-one by definition. ■

2 Onto

We say a function $f : A \to B$ is *"onto"* or *surjective* if every element in B corresponds to some element in A. Formally, $f : A \to B$ is surjective if and only if $\forall b \in B \exists a \in A | f(a) = b$. In other words, there are no "leftovers" in the codomain. (Again, refer to figures 3, 4, and 5 on pages 59-61 of the textbook.) Consider the following examples:

- $f : Cities \to States$ defined by $f(x) =$ location of x is surjective because every state contains at least one city. Notice that each state corresponds to may cities. This is acceptable.
- $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x + 1$ is surjective. For every $y \in \mathbb{R}$, we can find an $x \in \mathbb{R}$ (namely $x = (y - 1)$) such that $f(x) = y$. (In our case $f(x) = f(y - 1) = (y - 1) + 1 = y$)

[2] Be sure A is not empty!

[3] This means that we "defined" x and y to be elements of the positive real numbers.

- $f: States \to Cities$ defined by $f(x) =$capital of x is not surjective because there are many cities that are not the capital city of any state. For example, Detroit.
- $f: \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$ is not surjective because there is no real number such that $f(x) = -1$.

2-a Proving f is Surjective

To prove that $f: A \to B$ is surjective, we must demonstrate that for every $b \in B$, there exists an $a \in A$ such that $f(a) = b$. When dealing a specific function, we can often do this by constructing an a based on an arbitrary b. In other words, given b we find a "formula" for a. However, because b is an arbitrary element, finding this formula is not always easy. (A good way to obtain extra information about b is to use cases.)

2-a-i Proof Outline

In summary, to prove $f: A \to B$ is surjective:

1. Let b be an arbitrary element of B.
2. Prove that there exists an $a \in A$ such that $f(a) = b$. There are two possible approaches:
   - Construct a based on b. In other words, find a "formula" for a based on b.
   - Simply show that some a must exist. This is more difficult, but is sometimes necessary.
3. State that because there exists an a for an arbitrary b, then there exists an a for every b.
4. Conclude f is surjective.

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2-a-ii Example Proof

We will prove that $f: \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 3x + 7$ is surjective.

Let y be an arbitrary real number. Consider $x = \frac{y-7}{3}$. Because the real numbers are closed under addition and multiplication,[4] we know that x is in the domain, (i.e. $x \in \mathbb{R}$). Furthermore,

$$f(x) = f(\frac{y-7}{3}) = 3\frac{y-7}{3} + 7 = y$$

Thus, we can see that for any y we can find an x such that $f(x) = y$. Hence f is surjective by definition. ■

[4] In this case, multiplication by $\frac{1}{3}$.

3 Bijective and Inverse

Definition. $f : A \to B$ and $g : B \to A$ are *inverses* if and only if: $\forall a \in A$, $g(f(a)) = a$ and $\forall b \in b$, $f(g(b)) = b$. Intuitively, g is an inverse of f if g "reverses" the change made by f.

Definition. $f : A \to B$ is *bijective* if and only if it is injective and surjective.

Most proofs that f is bijective will contain two separate parts: A proof that f is injective, and a proof that f is bijective.

3-a Proof using Bijective and Inverse

Theorem: $f : A \to B$ has an inverse if and only if f is bijective.

3-a-i Outline

This proof has several parts. We begin with an outline:

1. If f has an inverse, then f is bijective.
   (a) If f has an inverse, then f is injective.
   (b) If f has an inverse, then f is surjective.
2. If f is bijective, then f has an inverse.

First, to prove a statement of the form "a if and only if b", we must prove both "if a, then b" and "if b, then a". This gives us steps 1 and 2. Next, to show that f is bijective, we must show that f is injective and that f is surjective. Thus, we break step 1 into steps 1a and 1b.

We now sketch a proof of each step:

- **1a:** We will do this by contradiction. We will assume f has an inverse g *and* assume to the contrary that f is not injective. Then we will demonstrate why it is impossible for f to not be injective.
- **1b:** We will prove this directly. We will assume that f has an inverse g. Then we will choose an arbitrary element $b \in B$ and construct an $a \in A$ such that $f(a) = b$.
- **2:** We will prove this directly. We will assume that f is bijective. We will then construct a function $g : B \to A$ and prove that
  - g is a valid function from B to A.
  - g is an inverse of f.

3-a-ii Formal Proof

We will prove that $f : A \to B$ has an inverse if and only if f is bijective. To prove this, we must prove two things:

1. If f has an inverse, then f is bijective.
2. If f is bijective, then f has an inverse.

We begin by proving part 1: We assume f has an inverse. Let $g : B \to A$ be that inverse. To show that f is bijective we must show that f is both injective and surjective. We first prove by contradiction that f is injective. Assume to the contrary that f is not injective. Because we assume f is not injective, we know that there there exist $a_1, a_2 \in A$ such that $a_1 \neq a_2$ and $f(a_1) = f(a_2) = b$ (where $b \in B$). From the definition of "inverse", we know that $g(b) = a_1$. However, this means that $g(f(a_2)) = a_1$ which contradicts the fact that g is the inverse of f. Hence f must be injective.

Next we show that f is surjective: Let b be an arbitrary element of B and consider $g(b)$. From the definition of inverse, we know that $f(g(b)) = b$. Therefore, for an arbitrary element $b \in B$, we have found an element $a = g(b)$ such that $f(a) = b$. Thus; we know that every $b \in B$ corresponds to such an $a \in A$; hence f is surjective.

Because we have shown f to be both surjective and injective, we can conclude that f is bijective as desired.

We must now prove part 2. We will do this by constructing a function $g : B \to A$ then demonstrating that g is a valid function and, in fact, the inverse of f.

Define $G : B \to A$ as follows: $g(b) = a$ where a is the unique element of f such that $f(a) = b$. Because f is surjective, we know that such an a exists. Because f is injective, we know that it is unique (i.e. that there is only one possible value for $g(b)$. Therefore, g is a valid function from B to A.

Finally, we need only show that g is, in fact, the inverse of f. Let a be an arbitrary element of A. If we let $b = f(a)$, the definition of g tells us that $g(b) = g(f(a)) = a$ as desired. Likewise, let b be an arbitrary element of b. Let $a = g(b)$. Again by definition of g, we know that $f(a) = f(g(b)) = b$. Hence, g is an inverse of f.

We have now proven both 1 and 2, thereby proving that $f : A \to B$ has an inverse if and only if f is bijective. $\blacksquare$