

3.3 Inference

3.3.1 Direct Inference (Modus Ponens) and Proofs

We concluded our last section with a proof that the sum of even numbers is even. There were several crucial ingredients in that proof. One consisted of introducing symbols for members of the universe of even numbers. How did we know to start the proof that way? As usual, there is more than one possible answer to this question. In this case, our intuition was probably based on thinking about what an even number is, and realizing that the definition itself is essentially symbolic. (You may argue that an even number is just twice another number, and you would be right. Apparently there are no symbols in that definition. But they really are there; they are the phrases “even number” and “another number.” Since we all know algebra is easier with symbolic variables rather than words, we should recognize that it makes sense to use algebraic notation.)

We then used the definition of even numbers, and our previous parenthetical comment suggests that it was natural for us to use the definition symbolically. The definition tells us that if m is an even number, then there exists another integer i such that $m = 2i$. We combined this with the assumption that m is even to conclude that $m = 2i$. This is an example of using the principle of *direct inference* (called *modus ponens* in Latin, though we won't worry about Latin names in this course.) The principle says that from p and $p \Rightarrow q$ we may conclude q . This principle is a cornerstone of logical arguments. If you think about the truth table for $p \Rightarrow q$, you can view this rule as describing one row of this truth table.

We then used algebra to show that if $m = 2i$ and $n = 2j$ then there exists a k such that $m + n = 2k$ (our k was $i + j$). Then we concluded that $m + n = 2k$. Next we used the definition of even number again. We finally reached a grand conclusion that for all pairs of even numbers, their sum is even. Here is another rule of inference, one of the more difficult to describe. We introduced the variables m and n . We used only well-known consequences of the fact that they were in the universe of even numbers in our proof. Thus we felt justified in asserting that what we concluded about m and n is true for any pair of integers. We might say that we were treating m and n as generic members of our universe. Thus our rule of inference says that if we can prove a statement for a generic member of our universe, then we can conclude it is true for every member of our universe. Perhaps the reason this rule is hard to put into words is that it is not simply a description of a truth table, but is a principle that we use in order to prove universally quantified statements.

3.3.2 Rules of inference

We have seen the ingredients of a typical proof. What do we mean by a proof in general? A proof of a statement is a convincing argument that the statement is true. A proof consists of a sequence of statements, each of which is either a hypothesis (to be described in more detail in our description of rules of inference below), a generally accepted fact, or the result of one of the following rules of inference for compound statements. (Note that the first four rules below are in effect a description of the truth tables for compound statements.) Rule 5 says what we must do in order to write a proof of an “if and only if” statement. Rule 6 (exemplified in our discussion above) is the principle of logical reasoning we called *direct inference*. Rule 7 is the transitive law, one we could derive by truth table analysis. Rule 8 may be regarded as another description of

the truth table of $p \Rightarrow q$. Rules 9 and 10 specify what we mean by the truth of a quantified statement.

- 1 From an example x that does not satisfy $p(x)$, we may conclude $\neg p(x)$
- 2 From $p(x)$ and $q(x)$, we may conclude $p(x) \wedge q(x)$
- 3 From either $p(x)$ or $q(x)$, we may conclude $p(x) \vee q(x)$
- 4 From either $q(x)$ or $\neg p(x)$ we may conclude $p(x) \Rightarrow q(x)$.
- 5 From $p(x) \Rightarrow q(x)$ and $q(x) \Rightarrow p(x)$ we may conclude $p(x) \Leftrightarrow q(x)$.
- 6 From $p(x)$ and $p(x) \Rightarrow q(x)$ we may conclude $q(x)$
- 7 From $p(x) \Rightarrow q(x)$ and $q(x) \Rightarrow r(x)$ we may conclude $p(x) \Rightarrow r(x)$.
- 8 If we can derive $q(x)$ from the hypothesis that x satisfies $p(x)$, then we may conclude $p(x) \Rightarrow q(x)$
- 9 If we can derive $p(x)$ from the hypothesis that x is a (generic) member of our universe, we may conclude $\forall x p(x)$.
- 10 From an example of an x satisfying $p(x)$ we may conclude $\exists x p(x)$.

A number of our rules of inference are redundant. However they are useful. For example, we could rewrite our proof that the sum of even numbers is even as follows.

“Let m and n be integers. If m is even, then there is a k with $m = 2k$. If n is even, then there is a j with $n = 2j$. Thus if m is even and n is even, there is a k and j such that $m + n = 2k + 2j = 2(k + j)$. Thus if m is even and m is even, there is an integer $h = k + j$ such that $m + n = 2h$. Thus if m is even and n is even, $m + n$ is even.”

This kind of argument could always be used to circumvent the use of rule 8, so it is not required as a rule of inference, but because it permits us to avoid such unnecessarily complicated “nonsense” in our proofs, we are including it.

3.3-1 Prove that if m is even, then m^2 is even.

3.3-2 Show that “ p implies q ” is equivalent to “ $\neg q$ implies $\neg p$.”

3.3-3 Is “ p implies q ” equivalent to “ q implies p ?”

In Exercise 3.3-1 we can mimic the proof that the sum of even numbers is even. We let m be an even number, use the fact that it is $2i$ for some i , and observe that $m^2 = (2i)^2 = 2 \cdot (2i^2)$, which lets us conclude that m^2 is even.

3.3.3 Contrapositive rule of inference.

In Exercise 3.3-2 we saw that if we know that $\neg q \Rightarrow \neg p$, then we can conclude that $p \Rightarrow q$. To see what that is good for, consider the following example.

Prove that if n is a positive integer with $n^2 > 100$, then $n > 10$.

Proof: Suppose n is not greater than 10. Then since n is between 1 and 10, n^2 is between 1 and 100. Thus n^2 is not greater than 100. Therefore, if $n^2 > 100$, n must be greater than 10. \square

We could give an intuitive explanation of the reasoning (and doing so would be a good exercise), but we don't need to because we can see we are just using the result of Exercise 2.

We adopt the result of Exercise 2 as a rule of inference, called the *contrapositive* rule of inference.

11 From $\neg q \Rightarrow \neg p$ we may conclude $p \Rightarrow q$.

3.3.4 Proof by contradiction

Proof by contrapositive inference is an example of what we call *indirect inference*. We have actually seen another example indirect inference, the method of proof by contradiction. Recall that in our proof of Euclid's Division Theorem we began by assuming that the theorem was false. We then chose among the pairs of integers m, n such that $m \neq qn + r$ with $0 \leq r < n$ a pair with the smallest possible m . We then made some computations by which we proved that indeed there is an r with $0 \leq r < n$ such that $m = qn + r$. Thus we started out by assuming the theorem was false, and from that assumption we drew two conclusions that contradicted each other. Since all our reasoning, except for the assumption that the theorem was false, used accepted rules of inference, the only source of that contradiction was our assumption. Thus our assumption had to be incorrect. This leads us to another rule of inference, called the principle of *proof by contradiction* or the principle of *reduction to absurdity*.

12 If from assuming p and $\neg q$, we can derive both r and $\neg r$ for some statement r , then we may conclude $p \Rightarrow q$.

There can be many variations of proof by contradiction. For example, we may assume p is true and q is false, and from this derive the contradiction that p is false, as in the following example.

Prove that if $x^2 + x - 2 = 0$, then $x \neq 0$.

Proof: Suppose that $x^2 + x - 2 = 0$. Assume that $x = 0$. Then $x^2 + x - 2 = 0 + 0 - 2 = -2$. This contradicts $x^2 + x - 2 = 0$. Thus (by the principle of proof by contradiction), if $x^2 + x - 2 = 0$, then $x \neq 0$. \square

Here the statement q was identical to p , namely $x^2 + x - 2 = 0$.

On the other hand, we may instead assume p is true and q is false, and derive a contradiction of a known fact, as in the following example.

Prove that if $x^2 + x - 2 = 0$, then $x \neq 0$.

Proof: Suppose that $x^2 + x - 2 = 0$. Assume that $x = 0$. Then $x^2 + x - 2 = 0 + 0 - 2 = -2$. Thus $0 = -2$, a contradiction. Thus (by the principle of proof by contradiction), if $x^2 + x - 2 = 0$, then $x \neq 0$. \square

Here the statement q is the known fact that $0 \neq -2$.

Sometimes the statement q that appears in the principle of proof by contradiction is simply a statement that arises naturally as we are trying to construct our proof, as in the following example.

Prove that if $x^2 + x - 2 = 0$, then $x \neq 0$.

Proof: Suppose that $x^2 + x - 2 = 0$. Then $x^2 + x = 2$. Assume that $x = 0$. Then $x^2 + x = 0 + 0 = 0$. But this is a contradiction. Thus (by the principle of proof by contradiction), if $x^2 + x - 2 = 0$, then $x \neq 0$. \square

Here the statement q is “ $x^2 + x = 2$.”

Finally, if proof by contradiction seems to you not to be much different from proof by contraposition, you are right, as the example that follows shows.

Prove that if $x^2 + x - 2 = 0$, then $x \neq 0$.

Proof: Assume that $x = 0$. Then $x^2 + x - 2 = 0 + 0 - 2 = -2$, so that $x^2 + x - 2 \neq 0$. Thus (by the principle of proof by contraposition), if $x^2 + x - 2 = 0$, then $x \neq 0$. \square

The last four examples illustrate the rich possibilities that indirect proof provides us. Of course they also illustrate why indirect proof can be confusing. There is no set formula that we use in writing a proof by contradiction, so there is no rule we can memorize in order to formulate indirect proofs. Instead, we have to ask ourselves whether assuming the opposite of what we are trying to prove gives us insight into why the assumption makes no sense. If it does, we have the basis of an indirect proof, and the way in which we choose to write it is a matter of personal choice.

3.3-4 Prove that if n is a positive integer such that $n^2 < 9$, then $n < 3$.

3.3-5 Prove that $\sqrt{5}$ is not rational.

Exercises

E3.3-1 Write down the converse and contrapositive of each of these statements.

1. If the hose is 60 feet long, then the hose will reach the tomatoes.
2. George goes for a walk only if Mary goes for a walk.
3. Pamela recites a poem if Andre asks for poem.

E3.3-2 Construct a proof that if n is even and n is odd, the $m + n$ is odd.

E3.3-3 Construct a proof that if m^2 is even, then m is even.

E3.3-4 Prove or disprove the following statement. “For every positive integer n , if n is prime, then 12 and $n^3 - n^2 + n$ have a common factor.”

E3.3-5 Prove or disprove the following statement. “For all integers b , c , and d , if x is a rational number such that $x^2 + bx + c = d$, then x is an integer.” (Hints: Are all the quantifiers given explicitly? It is ok to use the quadratic formula.)

E3.3-6 Prove that there is no largest prime number.