DISCRETE MATHEMATICS

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Chapter 1

LOGIC AND SETS

1.1. Sentences

In this section, we look at sentences, their truth or falsity, and ways of combining or connecting sentences to produce new sentences.

A sentence (or proposition) is an expression which is either true or false. The sentence "$2 + 2 = 4$" is true, while the sentence "π is rational" is false. It is, however, not the task of logic to decide whether any particular sentence is true or false. In fact, there are many sentences whose truth or falsity nobody has yet managed to establish; for example, the famous Goldbach conjecture that "every even number greater than 2 is a sum of two primes".

There is a defect in our definition. It is sometimes very difficult, under our definition, to determine whether or not a given expression is a sentence. Consider, for example, the expression "I am telling a lie"; am I?

Since there are expressions which are sentences under our definition, we proceed to discuss ways of connecting sentences to form new sentences.

Let p and q denote sentences.

DEFINITION. (CONJUNCTION) We say that the sentence $p \wedge q$ (p and q) is true if the two sentences p, q are both true, and is false otherwise.

EXAMPLE 1.1.1. The sentence "$2 + 2 = 4$ and $2 + 3 = 5$" is true.

EXAMPLE 1.1.2. The sentence "$2 + 2 = 4$ and π is rational" is false.

DEFINITION. (DISJUNCTION) We say that the sentence $p \vee q$ (p or q) is true if at least one of two sentences p, q is true, and is false otherwise.

EXAMPLE 1.1.3. The sentence "$2 + 2 = 2$ or $1 + 3 = 5$" is false.

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† This chapter was first used in lectures given by the author at Imperial College, University of London, in 1982.

Example 1.1.4. The sentence "$2 + 2 = 4$ or π is rational" is true.

Remark. To prove that a sentence $p \vee q$ is true, we may assume that the sentence p is false and use this to deduce that the sentence q is true in this case. For if the sentence p is true, our argument is already complete, never mind the truth or falsity of the sentence q.

Definition. (NEGATION) We say that the sentence $\overline{p}$ (not p) is true if the sentence p is false, and is false if the sentence p is true.

Example 1.1.5. The negation of the sentence "$2 + 2 = 4$" is the sentence "$2 + 2 \neq 4$".

Example 1.1.6. The negation of the sentence "π is rational" is the sentence "π is irrational".

Definition. (CONDITIONAL) We say that the sentence $p \to q$ (if p, then q) is true if the sentence p is false or if the sentence q is true or both, and is false otherwise.

Remark. It is convenient to realize that the sentence $p \to q$ is false precisely when the sentence p is true and the sentence q is false. To understand this, note that if we draw a false conclusion from a true assumption, then our argument must be faulty. On the other hand, if our assumption is false or if our conclusion is true, then our argument may still be acceptable.

Example 1.1.7. The sentence "if $2 + 2 = 2$, then $1 + 3 = 5$" is true, because the sentence "$2 + 2 = 2$" is false.

Example 1.1.8. The sentence "if $2 + 2 = 4$, then π is rational" is false.

Example 1.1.9. The sentence "if π is rational, then $2 + 2 = 4$" is true.

Definition. (DOUBLE CONDITIONAL) We say that the sentence $p \leftrightarrow q$ (p if and only if q) is true if the two sentences p, q are both true or both false, and is false otherwise.

Example 1.1.10. The sentence "$2 + 2 = 4$ if and only if π is irrational" is true.

Example 1.1.11. The sentence "$2 + 2 \neq 4$ if and only if π is rational" is also true.

If we use the letter T to denote "true" and the letter F to denote "false", then the above five definitions can be summarized in the following "truth table":

p	q	$p \wedge q$	$p \vee q$	$\overline{p}$	$p \to q$	$p \leftrightarrow q$
T	T	T	T	F	T	T
T	F	F	T	F	F	F
F	T	F	T	T	T	F
F	F	F	F	T	T	T

Remark. Note that in logic, "or" can mean "both". If you ask a logician whether he likes tea or coffee, do not be surprised if he wants both!

Example 1.1.12. The sentence $(p \vee q) \wedge \overline{(p \wedge q)}$ is true if exactly one of the two sentences p, q is true, and is false otherwise; we have the following "truth table":

p	q	$p \wedge q$	$p \vee q$	$\overline{p \wedge q}$	$(p \vee q) \wedge \overline{(p \wedge q)}$
T	T	T	T	F	F
T	F	F	T	T	T
F	T	F	T	T	T
F	F	F	F	T	F

1.2. Tautologies and Logical Equivalence

DEFINITION. A tautology is a sentence which is true on logical ground only.

EXAMPLE 1.2.1. The sentences $(p \wedge (q \wedge r)) \leftrightarrow ((p \wedge q) \wedge r)$ and $(p \wedge q) \leftrightarrow (q \wedge p)$ are both tautologies. This enables us to generalize the definition of conjunction to more than two sentences, and write, for example, $p \wedge q \wedge r$ without causing any ambiguity.

EXAMPLE 1.2.2. The sentences $(p \vee (q \vee r)) \leftrightarrow ((p \vee q) \vee r)$ and $(p \vee q) \leftrightarrow (q \vee p)$ are both tautologies. This enables us to generalize the definition of disjunction to more than two sentences, and write, for example, $p \vee q \vee r$ without causing any ambiguity.

EXAMPLE 1.2.3. The sentence $p \vee \overline{p}$ is a tautology.

EXAMPLE 1.2.4. The sentence $(p \rightarrow q) \leftrightarrow (\overline{q} \rightarrow \overline{p})$ is a tautology.

EXAMPLE 1.2.5. The sentence $(p \rightarrow q) \leftrightarrow (\overline{p} \vee q)$ is a tautology.

EXAMPLE 1.2.6. The sentence $\overline{(p \leftrightarrow q)} \leftrightarrow ((p \vee q) \wedge \overline{(p \wedge q)})$ is a tautology; we have the following "truth table":

p	q	$p \leftrightarrow q$	$\overline{(p \leftrightarrow q)}$	$(p \vee q) \wedge \overline{(p \wedge q)}$	$\overline{(p \leftrightarrow q)} \leftrightarrow ((p \vee q) \wedge \overline{(p \wedge q)})$
T	T	T	F	F	T
T	F	F	T	T	T
F	T	F	T	T	T
F	F	T	F	F	T

The following are tautologies which are commonly used. Let p, q and r denote sentences.

DISTRIBUTIVE LAW. *The following sentences are tautologies:*
(a) $(p \wedge (q \vee r)) \leftrightarrow ((p \wedge q) \vee (p \wedge r))$;
(b) $(p \vee (q \wedge r)) \leftrightarrow ((p \vee q) \wedge (p \vee r))$.

DE MORGAN LAW. *The following sentences are tautologies:*
(a) $\overline{(p \wedge q)} \leftrightarrow (\overline{p} \vee \overline{q})$;
(b) $\overline{(p \vee q)} \leftrightarrow (\overline{p} \wedge \overline{q})$.

INFERENCE LAW. *The following sentences are tautologies:*
(a) (MODUS PONENS) $(p \wedge (p \rightarrow q)) \rightarrow q$;
(b) (MODUS TOLLENS) $((p \rightarrow q) \wedge \overline{q}) \rightarrow \overline{p}$;
(c) (LAW OF SYLLOGISM) $((p \rightarrow q) \wedge (q \rightarrow r)) \rightarrow (p \rightarrow r)$.

These tautologies can all be demonstrated by truth tables. However, let us try to prove the first Distributive law here.

Suppose first of all that the sentence $p \wedge (q \vee r)$ is true. Then the two sentences p, $q \vee r$ are both true. Since the sentence $q \vee r$ is true, at least one of the two sentences q, r is true. Without loss of generality, assume that the sentence q is true. Then the sentence $p \wedge q$ is true. It follows that the sentence $(p \wedge q) \vee (p \wedge r)$ is true.

Suppose now that the sentence $(p \wedge q) \vee (p \wedge r)$ is true. Then at least one of the two sentences $(p \wedge q)$, $(p \wedge r)$ is true. Without loss of generality, assume that the sentence $p \wedge q$ is true. Then the two sentences p, q are both true. It follows that the sentence $q \vee r$ is true, and so the sentence $p \wedge (q \vee r)$ is true.

It now follows that the two sentences $p \wedge (q \vee r)$ and $(p \wedge q) \vee (p \wedge r)$ are either both true or both false, as the truth of one implies the truth of the other. It follows that the double conditional $(p \wedge (q \vee r)) \leftrightarrow ((p \wedge q) \vee (p \wedge r))$ is a tautology.

Definition. We say that two sentences p and q are logically equivalent if the sentence $p \leftrightarrow q$ is a tautology.

Example 1.2.7. The sentences $p \rightarrow q$ and $\overline{q} \rightarrow \overline{p}$ are logically equivalent. The latter is known as the contrapositive of the former.

Remark. The sentences $p \rightarrow q$ and $q \rightarrow p$ are **not** logically equivalent. The latter is known as the converse of the former.

1.3. Sentential Functions and Sets

In many instances, we have sentences, such as "x is even", which contains one or more variables. We shall call them sentential functions (or propositional functions).

Let us concentrate on our example "x is even". This sentence is true for certain values of x, and is false for others. Various questions arise:

(1) What values of x do we permit?
(2) Is the statement true for all such values of x in question?
(3) Is the statement true for some such values of x in question?

To answer the first of these questions, we need the notion of a universe. We therefore need to consider sets.

We shall treat the word "set" as a word whose meaning everybody knows. Sometimes we use the synonyms "class" or "collection". However, note that in some books, these words may have different meanings!

The important thing about a set is what it contains. In other words, what are its members? Does it have any?

If P is a set and x is an element of P, we write $x \in P$.

A set is usually described in one of the two following ways:

(1) by enumeration, *e.g.* $\{1, 2, 3\}$ denotes the set consisting of the numbers $1, 2, 3$ and nothing else;
(2) by a defining property (sentential function) $p(x)$. Here it is important to define a universe U to which all the x have to belong. We then write $P = \{x : x \in U \text{ and } p(x) \text{ is true}\}$ or, simply, $P = \{x : p(x)\}$.

The set with no elements is called the empty set and denoted by $\emptyset$.

Example 1.3.1. $\mathbb{N} = \{1, 2, 3, 4, 5, ...\}$ is called the set of natural numbers.

Example 1.3.2. $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$ is called the set of integers.

Example 1.3.3. $\{x : x \in \mathbb{N} \text{ and } -2 < x < 2\} = \{1\}$.

Example 1.3.4. $\{x : x \in \mathbb{Z} \text{ and } -2 < x < 2\} = \{-1, 0, 1\}$.

Example 1.3.5. $\{x : x \in \mathbb{N} \text{ and } -1 < x < 1\} = \emptyset$.

1.4. Set Functions

Suppose that the sentential functions $p(x)$, $q(x)$ are related to sets P, Q with respect to a given universe, *i.e.* $P = \{x : p(x)\}$ and $Q = \{x : q(x)\}$. We define

(1) the intersection $P \cap Q = \{x : p(x) \wedge q(x)\}$;
(2) the union $P \cup Q = \{x : p(x) \vee q(x)\}$;
(3) the complement $\overline{P} = \{x : \overline{p(x)}\}$; and
(4) the difference $P \setminus Q = \{x : p(x) \wedge \overline{q(x)}\}$.

The above are also sets. It is not difficult to see that
(1) $P \cap Q = \{x : x \in P \text{ and } x \in Q\}$;
(2) $P \cup Q = \{x : x \in P \text{ or } x \in Q\}$;
(3) $\overline{P} = \{x : x \notin P\}$; and
(4) $P \setminus Q = \{x : x \in P \text{ and } x \notin Q\}$.

We say that the set P is a subset of the set Q, denoted by $P \subseteq Q$ or by $Q \supseteq P$, if every element of P is an element of Q. In other words, if we have $P = \{x : p(x)\}$ and $Q = \{x : q(x)\}$ with respect to some universe U, then we have $P \subseteq Q$ if and only if the sentence $p(x) \to q(x)$ is true for all $x \in U$.

We say that two sets P and Q are equal, denoted by $P = Q$, if they contain the same elements, *i.e.* if each is a subset of the other, *i.e.* if $P \subseteq Q$ and $Q \subseteq P$.

Furthermore, we say that P is a proper subset of Q, denoted by $P \subset Q$ or by $Q \supset P$, if $P \subseteq Q$ and $P \neq Q$.

The following results on set functions can be deduced from their analogues in logic.

DISTRIBUTIVE LAW. *If P, Q, R are sets, then*
(a) $P \cap (Q \cup R) = (P \cap Q) \cup (P \cap R)$;
(b) $P \cup (Q \cap R) = (P \cup Q) \cap (P \cup R)$.

DE MORGAN LAW. *If P, Q are sets, then with respect to a universe U,*
(a) $\overline{(P \cap Q)} = \overline{P} \cup \overline{Q}$;
(b) $\overline{(P \cup Q)} = \overline{P} \cap \overline{Q}$.

We now try to deduce the first Distributive law for set functions from the first Distributive law for sentential functions.

Suppose that the sentential functions $p(x)$, $q(x)$, $r(x)$ are related to sets P, Q, R with respect to a given universe, *i.e.* $P = \{x : p(x)\}$, $Q = \{x : q(x)\}$ and $R = \{x : r(x)\}$. Then

$$P \cap (Q \cup R) = \{x : p(x) \wedge (q(x) \vee r(x))\}$$

and

$$(P \cap Q) \cup (P \cap R) = \{x : (p(x) \wedge q(x)) \vee (p(x) \wedge r(x))\}.$$

Suppose that $x \in P \cap (Q \cup R)$. Then $p(x) \wedge (q(x) \vee r(x))$ is true. By the first Distributive law for sentential functions, we have that

$$(p(x) \wedge (q(x) \vee r(x))) \leftrightarrow ((p(x) \wedge q(x)) \vee (p(x) \wedge r(x)))$$

is a tautology. It follows that $(p(x) \wedge q(x)) \vee (p(x) \wedge r(x))$ is true, so that $x \in (P \cap Q) \cup (P \cap R)$. This gives

$$P \cap (Q \cup R) \subseteq (P \cap Q) \cup (P \cap R). \tag{1}$$

Suppose now that $x \in (P \cap Q) \cup (P \cap R)$. Then $(p(x) \wedge q(x)) \vee (p(x) \wedge r(x))$ is true. It follows from the first Distributive law for sentential functions that $p(x) \wedge (q(x) \vee r(x))$ is true, so that $x \in P \cap (Q \cup R)$. This gives

$$(P \cap Q) \cup (P \cap R) \subseteq P \cap (Q \cup R). \tag{2}$$

The result now follows on combining (1) and (2).

1.5. Quantifier Logic

Let us return to the example "x is even" at the beginning of Section 1.3.

Suppose now that we restrict x to lie in the set $\mathbb{Z}$ of all integers. Then the sentence "x is even" is only true for some x in $\mathbb{Z}$. It follows that the sentence "some $x \in \mathbb{Z}$ are even" is true, while the sentence "all $x \in \mathbb{Z}$ are even" is false.

In general, consider a sentential function of the form $p(x)$, where the variable x lies in some clearly stated set. We can then consider the following two sentences:
(1) $\forall x,\ p(x)$ (for all x, $p(x)$ is true); and
(2) $\exists x,\ p(x)$ (for some x, $p(x)$ is true).

Definition. The symbols $\forall$ (for all) and $\exists$ (for some) are called the universal quantifier and the existential quantifier respectively.

Note that the variable x is a "dummy variable". There is no difference between writing $\forall x,\ p(x)$ or $\forall y,\ p(y)$.

Example 1.5.1. (LAGRANGE'S THEOREM) Every natural number is the sum of the squares of four integers. This can be written, in logical notation, as

$$\forall n \in \mathbb{N},\ \exists a,b,c,d \in \mathbb{Z},\ n = a^2 + b^2 + c^2 + d^2.$$

Example 1.5.2. (GOLDBACH CONJECTURE) Every even natural number greater than 2 is the sum of two primes. This can be written, in logical notation, as

$$\forall n \in \mathbb{N} \setminus \{1\},\ \exists p,q \text{ prime},\ 2n = p + q.$$

It is not yet known whether this is true or not. This is one of the greatest unsolved problems in mathematics.

1.6. Negation

Our main concern is to develop a rule for negating sentences with quantifiers. Let me start by saying that you are all fools. Naturally, you will disagree, and some of you will complain. So it is natural to suspect that the negation of the sentence $\forall x,\ p(x)$ is the sentence $\exists x,\ \overline{p(x)}$.

There is another way to look at this. Let U be the universe for all the x. Let $P = \{x : p(x)\}$. Suppose first of all that the sentence $\forall x,\ p(x)$ is true. Then $P = U$, so $\overline{P} = \emptyset$. But $\overline{P} = \{x : \overline{p(x)}\}$, so that if the sentence $\exists x,\ \overline{p(x)}$ were true, then $\overline{P} \neq \emptyset$, a contradiction. On the other hand, suppose now that the sentence $\forall x,\ p(x)$ is false. Then $P \neq U$, so that $\overline{P} \neq \emptyset$. It follows that the sentence $\exists x,\ \overline{p(x)}$ is true.

Now let me moderate a bit and say that some of you are fools. You will still complain, so perhaps none of you are fools. It is then natural to suspect that the negation of the sentence $\exists x,\ p(x)$ is the sentence $\forall x,\ \overline{p(x)}$.

To summarize, we simply "change the quantifier to the other type and negate the sentential function".

Suppose now that we have something more complicated. Let us apply bit by bit our simple rule. For example, the negation of

$$\forall x,\ \exists y,\ \forall z,\ \forall w,\ p(x,y,z,w)$$

is

$$\exists x,\ \overline{(\exists y,\ \forall z,\ \forall w,\ p(x,y,z,w))},$$

which is

$$\exists x,\ \forall y,\ \overline{(\forall z,\ \forall w,\ p(x,y,z,w))},$$

which is

$$\exists x,\ \forall y,\ \exists z,\ \overline{(\forall w,\ p(x,y,z,w))},$$

which is

$$\exists x,\ \forall y,\ \exists z,\ \exists w,\ \overline{p(x,y,z,w)}.$$

It is clear that the rule is the following: Keep the variables in their original order. Then, alter all the quantifiers. Finally, negate the sentential function.

EXAMPLE 1.6.1. The negation of the Goldbach conjecture is, in logical notation,

$$\exists n \in \mathbb{N} \setminus \{1\}, \ \forall p, q \text{ prime}, \ 2n \neq p + q.$$

In other words, there is an even natural number greater than 2 which is not the sum of two primes. In summary, to disprove the Goldbach conjecture, we simply need one counterexample!

PROBLEMS FOR CHAPTER 1

1. Using truth tables or otherwise, check that each of the following is a tautology:
   a) $p \to (p \vee q)$
   b) $((p \wedge \overline{q}) \to q) \to (p \to q)$
   c) $p \to (q \to p)$
   d) $(p \vee (p \wedge q)) \leftrightarrow p$
   e) $(p \to q) \leftrightarrow (\overline{q} \to \overline{p})$

2. Decide (and justify) whether each of the following is a tautology:
   a) $(p \vee q) \to (q \to (p \wedge q))$
   b) $(p \to (q \to r)) \to ((p \to q) \to (p \to r))$
   c) $((p \vee q) \wedge r) \leftrightarrow (p \vee (q \wedge r))$
   d) $(p \wedge q \wedge r) \to (s \vee t)$
   e) $(p \wedge q) \to (p \to q)$
   f) $\overline{p \to q} \leftrightarrow (\overline{p} \to \overline{q})$
   g) $(p \wedge \overline{q \wedge r}) \leftrightarrow (\overline{p \to q} \vee (p \wedge \overline{r}))$
   h) $((r \vee s) \to (p \wedge q)) \to (p \to (q \to (r \vee s)))$
   i) $p \to (q \wedge (r \vee s))$
   j) $(\overline{p \to q} \wedge (r \leftrightarrow s)) \to (t \to u)$
   k) $\overline{(p \wedge q) \vee r} \leftrightarrow ((\overline{p} \vee \overline{q}) \wedge \overline{r})$
   l) $(p \leftarrow q) \leftarrow (q \leftarrow p)$.
   m) $(p \wedge (q \vee (r \wedge s))) \leftrightarrow ((p \wedge q) \vee (p \wedge r \wedge s))$

3. For each of the following, decide whether the statement is true or false, and justify your assertion:
   a) If p is true and q is false, then $p \wedge q$ is true.
   b) If p is true, q is false and r is false, then $p \vee (q \wedge r)$ is true.
   c) The sentence $(p \leftrightarrow q) \leftrightarrow (\overline{q} \leftrightarrow \overline{p})$ is a tautology.
   d) The sentences $p \wedge (q \vee r)$ and $(p \vee q) \wedge (p \vee r)$ are logically equivalent.

4. List the elements of each of the following sets:
   a) $\{x \in \mathbb{N} : x^2 < 45\}$
   b) $\{x \in \mathbb{Z} : x^2 < 45\}$
   c) $\{x \in \mathbb{R} : x^2 + 2x = 0\}$
   d) $\{x \in \mathbb{Q} : x^2 + 4 = 6\}$
   e) $\{x \in \mathbb{Z} : x^4 = 1\}$
   f) $\{x \in \mathbb{N} : x^4 = 1\}$

5. How many elements are there in each of the following sets? Are the sets all different?
   a) $\emptyset$
   b) $\{\emptyset\}$
   c) $\{\{\emptyset\}\}$
   d) $\{\emptyset, \{\emptyset\}\}$
   e) $\{\emptyset, \emptyset\}$

6. Let $U = \{a, b, c, d\}$, $P = \{a, b\}$ and $Q = \{a, c, d\}$. Write down the elements of the following sets:
   a) $P \cup Q$
   b) $P \cap Q$
   c) $\overline{P}$
   d) $\overline{Q}$

7. Let $U = \mathbb{R}$, $A = \{x \in \mathbb{R} : x > 0\}$, $B = \{x \in \mathbb{R} : x > 1\}$ and $C = \{x \in \mathbb{R} : x < 2\}$. Find each of the following sets:
   a) $A \cup B$
   b) $A \cup C$
   c) $B \cup C$
   d) $A \cap B$
   e) $A \cap C$
   f) $B \cap C$
   g) $\overline{A}$
   h) $\overline{B}$
   i) $\overline{C}$
   j) $A \setminus B$
   k) $B \setminus C$

8. List all the subsets of the set $\{1, 2, 3\}$. How many subsets are there?

9. A, B, C, D are sets such that $A \cup B = C \cup D$, and both $A \cap B$ and $C \cap D$ are empty.
   a) Show by examples that $A \cap C$ and $B \cap D$ can be empty.
   b) Show that if $C \subseteq A$, then $B \subseteq D$.

10. Suppose that P, Q and R are subsets of $\mathbb{N}$. For each of the following, state whether or not the statement is true, and justify your assertion by studying the analogous sentences in logic:
    a) $P \cup (Q \cap R) = (P \cup Q) \cap (P \cup R)$.
    b) $P \subseteq Q$ if and only if $Q \subseteq P$.
    c) If $P \subseteq Q$ and $Q \subseteq R$, then $P \subseteq R$.

11. For each of the following sentences, write down the sentence in logical notation, negate the sentence, and say whether the sentence or its negation is true:
    a) Given any integer, there is a larger integer.
    b) There is an integer greater than all other integers.
    c) Every even number is a sum of two odd numbers.
    d) Every odd number is a sum of two even numbers.
    e) The distance between any two complex numbers is positive.
    f) All natural numbers divisible by 2 and by 3 are divisible by 6.
       [*Notation*: Write $x \mid y$ if x divides y.]
    g) Every integer is a sum of the squares of two integers.
    h) There is no greatest natural number.

12. For each of the following sentences, express the sentence in words, negate the sentence, and say whether the sentence or its negation is true:
    a) $\forall z \in \mathbb{N},\ z^2 \in \mathbb{N}$
    b) $\forall x \in \mathbb{Z},\ \forall y \in \mathbb{Z},\ \exists z \in \mathbb{Z},\ z^2 = x^2 + y^2$
    c) $\forall x \in \mathbb{Z},\ \forall y \in \mathbb{Z},\ (x > y) \to (x \neq y)$
    d) $\forall x, y, z \in \mathbb{R},\ \exists w \in \mathbb{R},\ x^2 + y^2 + z^2 = 8w$

13. Let $p(x, y)$ be a sentential function with variables x and y. Discuss whether each of the following is true on logical grounds only:
    a) $(\exists x,\ \forall y,\ p(x, y)) \to (\forall y,\ \exists x,\ p(x, y))$
    b) $(\forall y,\ \exists x,\ p(x, y)) \to (\exists x,\ \forall y,\ p(x, y))$

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