Permutations

Definition: A *permutation* of a set of distinct objects is an ordered arrangement of these objects.

Example: A permutation of objects

■	▲	●	★	
(1)	(2)	(3)	(4)	← object numbers

▲	■	★	●	
(2)	(1)	(4)	(3)	← object numbers
1	2	3	4	← position numbers

Object	2	is in position	1.
"	1	"	2.
"	4	"	3.
"	3	"	4.

Example: A permutation of a string

a	b	c	d	e	f
⋮	⋮	⋮	⋮	⋮	⋮
(1)	(2)	(3)	(4)	(5)	(6)
⋮	⋮	⋮	⋮	⋮	⋮
e	f	b	c	a	d

"efbcad" is a permutation of "abcdef". *Order* is part of a permutation.

Example: Another permutation

$$\begin{array}{cccccl} a_1 & a_2 & a_3 & a_4 & a_5 & \\ a_3 & a_4 & a_1 & a_5 & a_2 & \leftarrow \text{permutation} \end{array}$$

Definition: An r*-permutation* is an ordered arrangement of r elements in the set.

Example: Let $S = \{1, 2, 3, 4\}$

- (4,2,3,1) is a permutation of S
- (3,1,2) is a 3-permutation of S
- (4,3) is a 2-permutation of S

Notation: $P(n, r)$ is the number of r-permutations of a set of n elements.

What is the value of $P(n, r)$?

- There are r positions to be filled.

$$\begin{array}{ccccc} \sqcup & \sqcup & \sqcup & \cdots & \sqcup \\ 1 & 2 & 3 & & r \end{array}$$

- The first position can be filled in n ways.
- The second position can be filled in $n - 1$ ways.
- The ith position can be filled in $n - i + 1$ ways.

$$P(n, r) = \prod_{i=1}^{r}(n - i + 1) = n \cdot (n - 1) \cdots (n - r + 1) = \frac{n!}{(n - r)!}$$

Example: There are 6 contestants in an event. There are 3 prizes: a car, a TV, and a microwave. What is the number of distinct ways of awarding these prizes?

Number of such ways $= P(6, 3) = 6 \cdot 5 \cdot 4 = 120$.

Example: A salesperson has to visit 10 different cities. What is the number of possible orders for visiting these cities if they can be visited in any possible order?

$$\text{Ten positions:} \quad \begin{array}{ccccc} \sqcup & \sqcup & \sqcup & \cdots & \sqcup \\ 1 & 2 & 3 & \cdots & 10 \end{array}$$

If you fill in the positions from 1 to 10, then the ith position may be filled in $10 - i + 1$ ways, since the remaining cities at any step may be visited in any order.

\# of possible orders $= 10 \cdot 9 \cdot 8 \cdots 1 = 10!$

Combinations

Definition: An r-*combination* of elements of a set is an unordered selection of r elements from the set. An r-combination is just a subset of size r.

Example: Let $S = \{1, 2, 3, 4, 5\}$

- $\{1, 3\}$ is a 2-combination (a subset of size 2)
- $\{1, 3, 4, 5\}$ is a 4-combination

Notation: $C(n, r)$ is the number of r-combinations of a set of size n. I.e. the number of ways of choosing r objects out of n.

What is the value of $C(n, r)$?

- An r-permutation is obtained by first selecting a subset of r objects, and then permuting these r objects (i.e. placing an order on them).
- $P(n, r) = C(n, r) \cdot P(r, r)$, where $P(r, r)$ is the number of ways of ordering r objects.
- $C(n, r) = \frac{P(n,r)}{P(r,r)} = \frac{n!}{(n-r)!} \Big/ \frac{r!}{1!} = \frac{n!}{(n-r)!r!}$

$$\boxed{C(n, r) = C(n, n - r)}$$

Notation: $C(n, r)$ is also denoted as $\binom{n}{r}$.

$$\boxed{\binom{n}{r} = \binom{n}{n - r}}$$

The two previous boxed equations are true, because selecting r objects is equivalent to not selecting $n-r$ of them.

$$\begin{aligned} C(n,r) &= \frac{n!}{(n-r)!\,r!} \\ &= \frac{n!}{[n-(n-r)]!\,(n-r)!} \\ &= C(n,n-r) \end{aligned}$$

<u>Example</u>: There are 6 students in a class. In how many ways can we pick a squash team of 3 players?

$$\#\text{ of ways} = \binom{6}{3} = \frac{6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1}{(3\cdot 2\cdot 1)(3\cdot 2\cdot 1)} = 20$$

<u>Example</u>: A bin contains 10 red objects and 6 blue objects.

(i) What is the number of ways of picking 3 blue and 4 red objects?

(ii) What is the number of ways of picking the same number of objects of each color?

<u>Part (i)</u>:

3 blue objects may be picked in $\binom{6}{3}$ ways.

4 red objects can be chosen in $\binom{10}{4}$ ways.

So 3 blue objects and 4 red objects may be picked in $\binom{6}{3}\cdot\binom{10}{4}$ ways.

<u>Part (ii)</u>:

We can pick i objects of each color, where $i = 0, 1, \ldots 6$ (Six is the maximum, since there are only six blue balls).

$\#$ of ways of picking i blue and i red objects $= \binom{6}{i}\cdot\binom{10}{i}$

$$\begin{aligned} \text{Total \# of ways} &= \binom{6}{6}\cdot\binom{10}{6} + \binom{6}{5}\cdot\binom{10}{5} + \cdots + \binom{6}{0}\cdot\binom{10}{0} \\ &= \sum_{i=0}^{6}\left[\binom{6}{i}\cdot\binom{10}{i}\right] \end{aligned}$$

Pascal's Identity

$$C(n+1,k) = C(n,k-1) + C(n,k)$$

Proof idea: Let S be a set containing $n+1$ elements. Let a be some element in S, and let $T = S \setminus \{a\}$ (i.e. the set of all elements in S other than a). So we have $S = T \cup \{a\}$ and $|T| = n$. Then, a k-subset of S may be obtained by:

(i) picking a to be in the subset, and then picking $k-1$ more elements from T. There are $C(n,k-1)$ subsets of this type.

(ii) picking all k elements from T. There are $C(n,k)$ subsets of this type.

Vandermonde's Identity

$$C(m+n,r) = \sum_{k=0}^{r} [C(m,k) \cdot C(n,r-k)]$$

Proof idea: Let S be a set of m elements, and let T be a set of n elements such that $S \cap T = \emptyset$ (the empty set).

- $C(m+n,r) = \#$ of r-subsets of $S \cup T$
- An r-subset of $S \cup T$ is obtained by picking k elements from S and the remaining $r-k$ elements from T, where k could be $0, 1, \ldots, r$. For a given k, the number of such ways is $C(m,k) \cdot C(n,r-k)$.
- Summing over k gives the identity.

Binomial Theorem

$$(a+b)^n = \sum_{i=0}^{n} \left[C(n,i) \cdot a^i b^{n-1} \right]$$

Proof idea: What is the coefficient of $a^i b^{n-i}$?

$$n \text{ terms} \left\{ \begin{array}{cccc} (a+b) & (a+b) & \cdots & (a+b) \\ 1 & 2 & \cdots & n \end{array} \right.$$

The $a^i b^{n-i}$ term is obtained by picking a from i terms and using b from the remaining $n-i$ terms. There are $C(n,i)$ ways to do that.

Example: What is the coefficient of $a^2 b$ in $(a+b)^3$?

There are three ways to choose a from two terms and b from one:

$$\binom{3}{2} = 3 \text{ ways} \left\{ \begin{array}{ccc} (a+b) & (a+b) & (a+b) \\ & & \\ a & a & b \\ a & b & a \\ b & a & a \end{array} \right.$$

$$(a+b)(a+b)(a+b) = a^3 + 3a^2 b + 3ab^2 + b^3$$