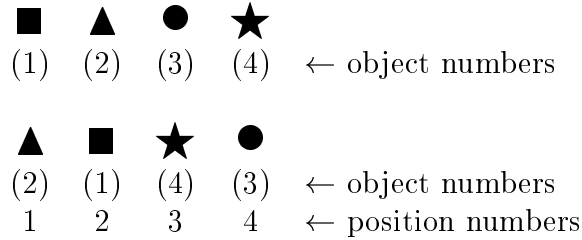


## Permutations

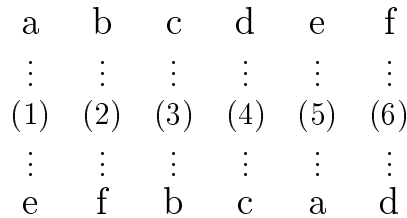
Definition: A *permutation* of a set of distinct objects is an ordered arrangement of these objects.

Example: A permutation of objects



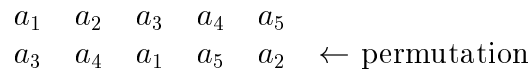
Object	2	is in position	1.
"	1	"	2.
"	4	"	3.
"	3	"	4.

Example: A permutation of a string



“efbcad” is a permutation of “abcdef”. *Order* is part of a permutation.

Example: Another permutation



Definition: An *r-permutation* is an ordered arrangement of  $r$  elements in the set.

Example: Let  $S = \{1, 2, 3, 4\}$

- (4,2,3,1) is a permutation of  $S$
- (3,1,2) is a 3-permutation of  $S$
- (4,3) is a 2-permutation of  $S$

Notation:  $P(n, r)$  is the number of  $r$ -permutations of a set of  $n$  elements.

What is the value of  $P(n, r)$ ?

- There are  $r$  positions to be filled.

$$\begin{array}{cccccc} \square & \square & \square & \cdots & \square \\ 1 & 2 & 3 & & r \end{array}$$

- The first position can be filled in  $n$  ways.
- The second position can be filled in  $n - 1$  ways.
- The  $i$ th position can be filled in  $n - i + 1$  ways.

$$P(n, r) = \prod_{i=1}^r (n - i + 1) = n \cdot (n - 1) \cdots (n - r + 1) = \frac{n!}{(n - r)!}$$

Example: There are 6 contestants in an event. There are 3 prizes: a car, a TV, and a microwave. What is the number of distinct ways of awarding these prizes?

$$\text{Number of such ways} = P(6, 3) = 6 \cdot 5 \cdot 4 = 120.$$

Example: A salesperson has to visit 10 different cities. What is the number of possible orders for visiting these cities if they can be visited in any possible order?

$$\begin{array}{cccccc} \text{Ten positions:} & \square & \square & \square & \cdots & \square \\ & 1 & 2 & 3 & \cdots & 10 \end{array}$$

If you fill in the positions from 1 to 10, then the  $i$ th position may be filled in  $10 - i + 1$  ways, since the remaining cities at any step may be visited in any order.

$$\# \text{ of possible orders} = 10 \cdot 9 \cdot 8 \cdots 1 = 10!$$

## Combinations

Definition: An  $r$ -combination of elements of a set is an unordered selection of  $r$  elements from the set. An  $r$ -combination is just a subset of size  $r$ .

Example: Let  $S = \{1, 2, 3, 4, 5\}$

- $\{1, 3\}$  is a 2-combination (a subset of size 2)
- $\{1, 3, 4, 5\}$  is a 4-combination

Notation:  $C(n, r)$  is the number of  $r$ -combinations of a set of size  $n$ . I.e. the number of ways of choosing  $r$  objects out of  $n$ .

What is the value of  $C(n, r)$ ?

- An  $r$ -permutation is obtained by first selecting a subset of  $r$  objects, and then permuting these  $r$  objects (i.e. placing an order on them).
- $P(n, r) = C(n, r) \cdot P(r, r)$ , where  $P(r, r)$  is the number of ways of ordering  $r$  objects.
- $C(n, r) = \frac{P(n, r)}{P(r, r)} = \frac{n!}{(n-r)!} \frac{1}{r!} = \frac{n!}{(n-r)!r!}$

$$C(n, r) = C(n, n - r)$$

Notation:  $C(n, r)$  is also denoted as  $\binom{n}{r}$ .

$$\binom{n}{r} = \binom{n}{n - r}$$

The two previous boxed equations are true, because selecting  $r$  objects is equivalent to not selecting  $n - r$  of them.

$$\begin{aligned} C(n, r) &= \frac{n!}{(n-r)! r!} \\ &= \frac{n!}{[n - (n-r)]! (n-r)!} \\ &= C(n, n-r) \end{aligned}$$

Example: There are 6 students in a class. In how many ways can we pick a squash team of 3 players?

$$\# \text{ of ways} = \binom{6}{3} = \frac{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{(3 \cdot 2 \cdot 1)(3 \cdot 2 \cdot 1)} = 20$$

Example: A bin contains 10 red objects and 6 blue objects.

- (i) What is the number of ways of picking 3 blue and 4 red objects?
- (ii) What is the number of ways of picking the same number of objects of each color?

Part (i):

3 blue objects may be picked in  $\binom{6}{3}$  ways.

4 red objects can be chosen in  $\binom{10}{4}$  ways.

So 3 blue objects and 4 red objects may be picked in  $\binom{6}{3} \cdot \binom{10}{4}$  ways.

Part (ii):

We can pick  $i$  objects of each color, where  $i = 0, 1, \dots, 6$  (Six is the maximum, since there are only six blue balls).

# of ways of picking  $i$  blue and  $i$  red objects =  $\binom{6}{i} \cdot \binom{10}{i}$

$$\begin{aligned} \text{Total \# of ways} &= \binom{6}{6} \cdot \binom{10}{6} + \binom{6}{5} \cdot \binom{10}{5} + \dots + \binom{6}{0} \cdot \binom{10}{0} \\ &= \sum_{i=0}^6 \left[ \binom{6}{i} \cdot \binom{10}{i} \right] \end{aligned}$$

## Pascal's Identity

$$C(n + 1, k) = C(n, k - 1) + C(n, k)$$

Proof idea: Let  $S$  be a set containing  $n + 1$  elements. Let  $a$  be some element in  $S$ , and let  $T = S \setminus \{a\}$  (i.e. the set of all elements in  $S$  other than  $a$ ). So we have  $S = T \cup \{a\}$  and  $|T| = n$ . Then, a  $k$ -subset of  $S$  may be obtained by:

- (i) picking  $a$  to be in the subset, and then picking  $k - 1$  more elements from  $T$ . There are  $C(n, k - 1)$  subsets of this type.
- (ii) picking all  $k$  elements from  $T$ . There are  $C(n, k)$  subsets of this type.

## Vandermonde's Identity

$$C(m + n, r) = \sum_{k=0}^r [C(m, k) \cdot C(n, r - k)]$$

Proof idea: Let  $S$  be a set of  $m$  elements, and let  $T$  be a set of  $n$  elements such that  $S \cap T = \emptyset$  (the empty set).

- $C(m + n, r) = \#$  of  $r$ -subsets of  $S \cup T$
- An  $r$ -subset of  $S \cup T$  is obtained by picking  $k$  elements from  $S$  and the remaining  $r - k$  elements from  $T$ , where  $k$  could be  $0, 1, \dots, r$ . For a given  $k$ , the number of such ways is  $C(m, k) \cdot C(n, r - k)$ .
- Summing over  $k$  gives the identity.

Binomial Theorem

$$(a + b)^n = \sum_{i=0}^n [C(n, i) \cdot a^i b^{n-i}]$$

Proof idea: What is the coefficient of  $a^i b^{n-i}$ ?

$$n \text{ terms } \left\{ \begin{array}{cccc} (a+b) & (a+b) & \cdots & (a+b) \\ 1 & 2 & \cdots & n \end{array} \right.$$

The  $a^i b^{n-i}$  term is obtained by picking  $a$  from  $i$  terms and using  $b$  from the remaining  $n - i$  terms. There are  $C(n, i)$  ways to do that.

Example: What is the coefficient of  $a^2 b$  in  $(a + b)^3$ ?

There are three ways to choose  $a$  from two terms and  $b$  from one:

$$\binom{3}{2} = 3 \text{ ways } \left\{ \begin{array}{ccc} (a+b) & (a+b) & (a+b) \\ a & a & b \\ a & b & a \\ b & a & a \end{array} \right.$$

$$(a + b)(a + b)(a + b) = a^3 + 3a^2b + 3ab^2 + b^3$$