MATHEMATICAL INDUCTION

Mathematical induction is a method used to prove stated results, equations or identities whenever it is very difficult to proceed by normal methods.
The procedure is quite simple but for the mathematical manipulations. Given a result to prove, we proceed as follows:

1. We show that the result is true for n = 1 and n = 2.
2. We assume that the result is true for $n = k$.
3. We prove that it is true for $n = k + 1$.

The logic behind this method is that, if it true for $n = k$ and proven true for $n = k + 1$, then it should be true for all integral values of n.
We shall now take some examples to illustrate this method.

EXAMPLE 1

Prove that $\sum_{r=1}^{n} r(r+3) = \frac{n(n+1)(n+5)}{3}$.

SOLUTION

When n = 1, L.H.S = $\sum_{r=1}^{1} r(r+3) = (1)(4) = 4$;   R.H.S = $\frac{(1)(2)(6)}{3} = 4$.

When n = 2, L.H.S = $\sum_{r=1}^{2} r(r+3) = (1)(4) + (2)(5) = 14$;   R.H.S = $\frac{(2)(3)(7)}{3} = 14.$.

Assume that the result is true for $n = k$, that is,

$$\sum_{r=1}^{k} r(r+3) = \frac{k(k+1)(k+5)}{3}$$

To prove that it is true for $n = k + 1$,

$$\sum_{r=1}^{k+1} r(r+3) = \sum_{r=1}^{k} r(r+3) + (k+1)(k+4) = \frac{k(k+1)(k+5)}{3} + (k+1)(k+4)$$

$$= \frac{(k+1)}{3}\left[k(k+5) + 3(k+4)\right] = \frac{(k+1)}{3}(k^2 + 8k + 12)$$

$$= \frac{(k+1)(k+2)(k+6)}{3} = \frac{[k+1][(k+1)+1][(k+1)+5]}{3}.$$

EXAMPLE 2

If $n \in Z^+$, prove that $7^n(3n+1)-1$ is always divisible by 9.

SOLUTION

Let $T_n = 7^n(3n+1)-1$;
$T_1 = (7)(4)-1 = 27$ is divisible by 9.
$T_2 = (49)(7)-1 = 342$ is divisible by 9.

Assume that the result is true for $n = k$, that is, $T_k = 7^k(3k+1)-1$ is divisible by 9.
To prove that the result is true for $n = k + 1$, that is, we have to prove that the difference between T_{k+1} and T_k is also divisible by 9 (think about it!).

Now, $T_{k+1} - T_k = [7^{k+1}(3k+4)-1]-[7^k(3k+1)-1] = 7^k(21k+28)-7^k(3k+1)$
$= 7^k(18k+27) = 9[7^k(2k+3)]$
which is obviously divisible by 9.

EXAMPLE 3

If $y = xe^x$, prove that $\frac{d^n y}{dx^n} = (x+n)\ e^x$.

SOLUTION

$\frac{dy}{dx} = xe^x + e^x = (x+1)\ e^x$ = R.H.S when n = 1.
$\frac{d^2 y}{dx^2} = \frac{d}{dx}(xe^x + e^x) = xe^x + e^x + e^x = (x+2)\ e^x$ = R.H.S when n = 2.

Assume that the result is true when $n = k$, that is, $\frac{d^k y}{dx^k} = (x+k)\ e^x$.
$\frac{d^{k+1} y}{dx^{k+1}} = \frac{d}{dx}\left(\frac{d^k y}{dx^k}\right) = (x+k)\ e^x + e^x$ = [x + (k + 1) e^x].

EXAMPLE 4

Given that $n \geq 1$, show that $5(4n^2 + 1) > 4(n+1)^2 + 1$.
Hence, or otherwise, prove by induction that $5^n \geq 4n^2 + 1$.

SOLUTION

The first part can easily be proved. We start with the result to be proved and, by sending all the terms on the L.H.S., we end up with a quadratic inequality in *n*. Solving for the range of n, we find that $n \geq \frac{2}{3}$ and this implies that the inequality is also valid for $n \geq 1$.

To prove that $5^n \geq 4n^2 + 1$;
When n = 1, L.H.S = 5 and R.H.S = 5.
When n = 2, L.H.S = 25 and R.H.S = 17.

Assume that $5^k \geq 4k^2 + 1$; to prove the case for $n = k + 1$, multiply by 5 on both sides, that is, $5(5^k) \geq 5(4k^2 + 1) \Rightarrow 5^{k+1} \geq 5(4k^2 + 1) > 4(k+1)^2 + 1$ (from the above result).
Therefore, $5^{k+1} > 4(k+1)^2 + 1$.

EXAMPLE 5

Prove that $(\cos q + i \sin q)^n = \cos nq + i \sin nq$.

SOLUTION

When n = 1, L.H.S = $\cos q + i \sin q$ = R.H.S
When n = 2, L.H.S = $(\cos q + i \sin q)^2 = \cos^2 q + 2i \cos q \sin q + \sin^2 q$

$$= (\cos^2 q + \sin^2 q) + i(2 \sin q \cos q)$$
$$= \cos 2q + i \sin 2q \text{ = R.H.S}$$

Assume that the result is true for $n = k$, that is, $(\cos q + i \sin q)^k = \cos kq + i \sin kq$;

$$\begin{aligned}(\cos q + i \sin q)^{k+1} &= (\cos q + i \sin q)^k (\cos q + i \sin q) \\ &= (\cos nq + i \sin nq)(\cos q + i \sin q) \\ &= (\cos nq \cos q - \sin nq \sin q) + i(\cos nq \sin q + \sin nq \cos q) \\ &= \cos(nq + q) + i \sin(nq + q) = \cos(n+1)q + i \sin(n+1)q\end{aligned}$$

EXAMPLE 6

Prove by induction that

$$\cos a + \cos(a+2h) + \cos(a+4h) + ... + \cos[a+(2n-2)h] = \cos[a+(n-1)h]\frac{\sin nh}{\sin h}$$

SOLUTION

When n = 1, L.H.S = cos a; R.H.S = $\cos a \dfrac{\sin h}{\sin h} = \cos a$.

When n = 2, L.H.S = cos a + cos (a + 2h);

$$\text{R.H.S} = \cos(a+h)\frac{\sin 2h}{\sin h} = \cos(a+h)\frac{2\sin h \cos h}{\sin h} = 2\cos(a+h)\cos h$$

= cos a + cos (a + 2h).

Assume that the result is true for $n = k$, that is,

$$\cos a + \cos(a+2h) + \cos(a+4h) + ... + \cos[a+(2k-2)h] = \cos[a+(k-1)h]\frac{\sin kh}{\sin h};$$

For $n = k + 1$,

$$\text{L.H.S} = \cos a + \cos(a+2h) + \cos(a+4h) + ... + \cos[a+(2k-2)h] + \cos(a+2kh)$$

$$= \cos[a+(k-1)h]\frac{\sin kh}{\sin h} + \cos(a+2kh)$$

$$= \frac{\cos[(a+kh)-h]\sin kh + \cos[(a+kh)+kh]\sin h}{\sin h}$$

$$= \frac{[\cos(a+kh)\cos h + \sin(a+kh)\sin h]\sin kh + [\cos(a+kh)\cos kh - \sin(a+kh)\sin kh]\sin h}{\sin h}$$

$$= \frac{\cos(a+kh)\cos h \sin kh + \cos(a+kh)\cos kh \sin h}{\sin h}$$

$$= \frac{\cos(a+kh)}{\sin h}\sin[(k+1)h].$$