DISCRETE MATHEMATICS

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Chapter 2

RELATIONS AND FUNCTIONS

2.1. Relations

We start by considering a simple example. Let S denote the set of all students at Macquarie University, and let T denote the set of all teaching staff here. For every student $s \in S$ and every teaching staff $t \in T$, exactly one of the following is true:

(1) s has attended a lecture given by t; or
(2) s has not attended a lecture given by t.

We now define a relation $\mathcal{R}$ as follows. Let $s \in S$ and $t \in T$. We say that $s\mathcal{R}t$ if s has attended a lecture given by t. If we now look at all possible pairs (s,t) of students and teaching staff, then some of these pairs will satisfy the relation while other pairs may not. To put it in a slightly different way, we can say that the relation $\mathcal{R}$ can be represented by the collection of all pairs (s,t) where $s\mathcal{R}t$. This is a subcollection of the set of all possible pairs (s,t).

Definition. Let A and B be sets. The set $A \times B = \{(a,b) : a \in A \text{ and } b \in B\}$ is called the cartesian product of the sets A and B. In other words, $A \times B$ is the set of all ordered pairs (a,b), where $a \in A$ and $b \in B$.

Definition. Let A and B be sets. By a relation $\mathcal{R}$ on A and B, we mean a subset of the cartesian product $A \times B$.

Remark. There are many instances when $A = B$. Then by a relation $\mathcal{R}$ on A, we mean a subset of the cartesian product $A \times A$.

Example 2.1.1. Let $A = \{1,2,3,4\}$. Define a relation $\mathcal{R}$ on A by writing $(x,y) \in \mathcal{R}$ if $x < y$. Then $\mathcal{R} = \{(1,2),(1,3),(1,4),(2,3),(2,4),(3,4)\}$.

† This chapter was first used in lectures given by the author at Imperial College, University of London, in 1982.

EXAMPLE 2.1.2. Let A be the power set of the set $\{1,2\}$; in other words, $A = \{\emptyset, \{1\}, \{2\}, \{1,2\}\}$ is the set of subsets of the set $\{1,2\}$. Then it is not too difficult to see that

$$\mathcal{R} = \{(\emptyset, \{1\}), (\emptyset, \{2\}), (\emptyset, \{1,2\}), (\{1\}, \{1,2\}), (\{2\}, \{1,2\})\}$$

is a relation on A where $(P, Q) \in \mathcal{R}$ if $P \subset Q$.

2.2. Equivalence Relations

We begin by considering a familiar example.

EXAMPLE 2.2.1. A rational number is a number of the form p/q, where $p \in \mathbb{Z}$ and $q \in \mathbb{N}$. This can also be viewed as an ordered pair (p, q), where $p \in \mathbb{Z}$ and $q \in \mathbb{N}$. Let $A = \{(p, q) : p \in \mathbb{Z} \text{ and } q \in \mathbb{N}\}$. We can define a relation $\mathcal{R}$ on A by writing $((p_1, q_1), (p_2, q_2)) \in \mathcal{R}$ if $p_1 q_2 = p_2 q_1$, *i.e.* if $p_1/q_1 = p_2/q_2$. This relation $\mathcal{R}$ has some rather interesting properties:
(1) $((p, q), (p, q)) \in \mathcal{R}$ for all $(p, q) \in A$;
(2) whenever $((p_1, q_1), (p_2, q_2)) \in \mathcal{R}$, we have $((p_2, q_2), (p_1, q_1)) \in \mathcal{R}$; and
(3) whenever $((p_1, q_1), (p_2, q_2)) \in \mathcal{R}$ and $((p_2, q_2), (p_3, q_3)) \in \mathcal{R}$, we have $((p_1, q_1), (p_3, q_3)) \in \mathcal{R}$.
This is usually known as the equivalence of fractions.

We now investigate these properties in a more general setting. Let $\mathcal{R}$ be a relation on a set A.

DEFINITION. Suppose that for all $a \in A$, $(a, a) \in \mathcal{R}$. Then we say that $\mathcal{R}$ is reflexive.

EXAMPLE 2.2.2. The relation $\mathcal{R}$ defined on the set $\mathbb{Z}$ by $(a, b) \in \mathcal{R}$ if $a \leq b$ is reflexive.

EXAMPLE 2.2.3. The relation $\mathcal{R}$ defined on the set $\mathbb{Z}$ by $(a, b) \in \mathcal{R}$ if $a < b$ is not reflexive.

DEFINITION. Suppose that for all $a, b \in A$, $(b, a) \in \mathcal{R}$ whenever $(a, b) \in \mathcal{R}$. Then we say that $\mathcal{R}$ is symmetric.

EXAMPLE 2.2.4. Let $A = \{1, 2, 3\}$.
(1) The relation $\mathcal{R} = \{(1, 2), (2, 1)\}$ is symmetric but not reflexive.
(2) The relation $\mathcal{R} = \{(1, 1), (2, 2), (3, 3)\}$ is reflexive and symmetric.
(3) The relation $\mathcal{R} = \{(1, 1), (2, 2), (3, 3), (1, 2)\}$ is reflexive but not symmetric.

DEFINITION. Suppose that for all $a, b, c \in A$, $(a, c) \in \mathcal{R}$ whenever $(a, b) \in \mathcal{R}$ and $(b, c) \in \mathcal{R}$. Then we say that $\mathcal{R}$ is transitive.

EXAMPLE 2.2.5. Let $A = \{1, 2, 3\}$.
(1) The relation $\mathcal{R} = \{(1, 2), (2, 1), (1, 1), (2, 2)\}$ is symmetric and transitive but not reflexive.
(2) The relation $\mathcal{R} = \{(1, 1), (2, 2), (3, 3)\}$ is reflexive, symmetric and transitive.
(3) The relation $\mathcal{R} = \{(1, 1), (2, 2), (3, 3), (1, 2)\}$ is reflexive and transitive but not symmetric.

DEFINITION. Suppose that a relation $\mathcal{R}$ on a set A is reflexive, symmetric and transitive. Then we say that $\mathcal{R}$ is an equivalence relation.

EXAMPLE 2.2.6. Define a relation $\mathcal{R}$ on $\mathbb{Z}$ by writing $(a, b) \in \mathcal{R}$ if the integer $a - b$ is a multiple of 3. Then $\mathcal{R}$ is an equivalence relation on $\mathbb{Z}$. To see this, note that for every $a \in \mathbb{Z}$, $a - a = 0$ is clearly a multiple of 3, so that $(a, a) \in \mathcal{R}$. It follows that $\mathcal{R}$ is reflexive. Suppose now that $a, b \in \mathbb{Z}$. If $(a, b) \in \mathcal{R}$, then $a - b$ is a multiple of 3. In other words, $a - b = 3k$ for some $k \in \mathbb{Z}$, so that $b - a = 3(-k)$. Hence $b - a$ is a multiple of 3, so that $(b, a) \in \mathcal{R}$. It follows that $\mathcal{R}$ is symmetric. Suppose now that $a, b, c \in \mathbb{Z}$. If $(a, b), (b, c) \in \mathcal{R}$, then $a - b$ and $b - c$ are both multiples of 3. In other words, $a - b = 3k$ and $b - c = 3m$ for some $k, m \in \mathbb{Z}$, so that $a - c = 3(k + m)$. Hence $a - c$ is a multiple of 3, so that $(a, c) \in \mathcal{R}$. It follows that $\mathcal{R}$ is transitive.

2.3. Equivalence Classes

Let us examine more carefully our last example, where the relation $\mathcal{R}$, where $(a,b) \in \mathcal{R}$ if the integer $a-b$ is a multiple of 3, is an equivalence relation on $\mathbb{Z}$. Note that the elements in the set

$$\{\ldots,-9,-6,-3,0,3,6,9,\ldots\}$$

are all related to each other, but not related to any integer outside this set. The same phenomenon applies to the sets

$$\{\ldots,-8,-5,-2,1,4,7,\ldots\} \qquad \text{and} \qquad \{\ldots,-7,-4,-1,2,5,8,\ldots\}.$$

In other words, the relation $\mathcal{R}$ has split the set $\mathbb{Z}$ into three disjoint parts.

In general, let $\mathcal{R}$ denote an equivalence relation on a set A.

DEFINITION. For every $a \in A$, the set $[a] = \{b \in A : (a,b) \in \mathcal{R}\}$ is called the equivalence class of A containing a.

LEMMA 2A. *Suppose that $\mathcal{R}$ is an equivalence relation on a set A, and that $a, b \in A$. Then $b \in [a]$ if and only if $[a] = [b]$.*

PROOF. ($\Rightarrow$) Suppose that $b \in [a]$. Then $(a,b) \in \mathcal{R}$. We shall show that $[a] = [b]$ by showing that (1) $[b] \subseteq [a]$ and (2) $[a] \subseteq [b]$. To show (1), let $c \in [b]$. Then $(b,c) \in \mathcal{R}$. It now follows from the transitive property that $(a,c) \in \mathcal{R}$, so that $c \in [a]$. (1) follows. To show (2), let $c \in [a]$. Then $(a,c) \in \mathcal{R}$. Since $(a,b) \in \mathcal{R}$, it follows from the symmetric property that $(b,a) \in \mathcal{R}$. It now follows from the transitive property that $(b,c) \in \mathcal{R}$, so that $c \in [b]$. (2) follows.

($\Leftarrow$) Suppose that $[a] = [b]$. By the reflexive property, $(b,b) \in \mathcal{R}$, so that $b \in [b]$, so that $b \in [a]$. ♣

LEMMA 2B. *Suppose that $\mathcal{R}$ is an equivalence relation on a set A, and that $a, b \in A$. Then either $[a] \cap [b] = \emptyset$ or $[a] = [b]$.*

PROOF. Suppose that $[a] \cap [b] \neq \emptyset$. Let $c \in [a] \cap [b]$. Then it follows from Lemma 2A that $[c] = [a]$ and $[c] = [b]$, so that $[a] = [b]$. ♣

We have therefore proved

THEOREM 2C. *Suppose that $\mathcal{R}$ is an equivalence relation on a set A. Then A is the disjoint union of its distinct equivalence classes.*

EXAMPLE 2.3.1. Let $m \in \mathbb{N}$. Define a relation on $\mathbb{Z}$ by writing $x \equiv y \pmod{m}$ if $x - y$ is a multiple of m. It is not difficult to check that this is an equivalence relation, and that $\mathbb{Z}$ is partitioned into the equivalence classes $[0], [1], \ldots, [m-1]$. These are called the residue (or congruence) classes modulo m, and the set of these m residue classes is denoted by $\mathbb{Z}_m$.

2.4. Functions

Let A and B be sets. A function (or mapping) f from A to B assigns to each $x \in A$ an element $f(x)$ in B. We write $f : A \to B : x \mapsto f(x)$ or simply $f : A \to B$. A is called the domain of f, and B is called the codomain of f. The element $f(x)$ is called the image of x under f. Furthermore, the set $f(B) = \{y \in B : y = f(x) \text{ for some } x \in A\}$ is called the range or image of f.

Two functions $f : A \to B$ and $g : A \to B$ are said to be equal, denoted by $f = g$, if $f(x) = g(x)$ for every $x \in A$.

It is sometimes convenient to express a function by its graph G. This is defined by

$$G = \{(x, f(x)) : x \in A\} = \{(x, y) : x \in A \text{ and } y = f(x) \in B\}.$$

EXAMPLE 2.4.1. Consider the function $f : \mathbb{N} \to \mathbb{N}$ defined by $f(x) = 2x$ for every $x \in \mathbb{N}$. Then the domain and codomain of f are $\mathbb{N}$, while the range of f is the set of all even natural numbers.

EXAMPLE 2.4.2. Consider the function $f : \mathbb{Z} \to \mathbb{Z} : x \mapsto |x|$. Then the domain and codomain of f are $\mathbb{Z}$, while the range of f is the set of all non-negative integers.

EXAMPLE 2.4.3. There are four functions from $\{a, b\}$ to $\{1, 2\}$.

EXAMPLE 2.4.4. Suppose that A and B are finite sets, with n and m elements respectively. An interesting question is to determine the number of different functions $f : A \to B$ that can be defined. Without loss of generality, let $A = \{1, 2, \ldots, n\}$. Then there are m different ways of choosing a value for $f(1)$ from the elements of B. For each such choice of $f(1)$, there are again m different ways of choosing a value for $f(2)$ from the elements of B. For each such choice of $f(1)$ and $f(2)$, there are again m different ways of choosing a value for $f(3)$ from the elements of B. And so on. It follows that the number of different functions $f : A \to B$ that can be defined is equal to the number of ways of choosing $(f(1), \ldots, f(n))$. The number of such ways is clearly

$$\underbrace{m \quad \ldots \quad m}_{n} = m^n.$$

Example 2.4.2 shows that a function can map different elements of the domain to the same element in the codomain. Also, the range of a function may not be all of the codomain.

DEFINITION. We say that a function $f : A \to B$ is one-to-one if $x_1 = x_2$ whenever $f(x_1) = f(x_2)$.

DEFINITION. We say that a function $f : A \to B$ is onto if for every $y \in B$, there exists $x \in A$ such that $f(x) = y$.

REMARKS. (1) If a function $f : A \to B$ is one-to-one and onto, then an inverse function exists. To see this, take any $y \in B$. Since the function $f : A \to B$ is onto, it follows that there exists $x \in A$ such that $f(x) = y$. Suppose now that $z \in A$ satisfies $f(z) = y$. Then since the function $f : A \to B$ is one-to-one, it follows that we must have $z = x$. In other words, there is precisely one $x \in A$ such that $f(x) = y$. We can therefore define an inverse function $f^{-1} : B \to A$ by writing $f^{-1}(y) = x$, where $x \in A$ is the unique solution of $f(x) = y$.

(2) Consider a function $f : A \to B$. Then f is onto if and only if for every $y \in B$, there is at least one $x \in A$ such that $f(x) = y$. On the other hand, f is one-to-one if and only if for every $y \in B$, there is at most one $x \in A$ such that $f(x) = y$.

EXAMPLE 2.4.5. Consider the function $f : \mathbb{N} \to \mathbb{N} : x \mapsto x$. This is one-to-one and onto.

EXAMPLE 2.4.6. Consider the function $f : \mathbb{N} \to \mathbb{Z} : x \mapsto x$. This is one-to-one but not onto.

EXAMPLE 2.4.7. Consider the function $f : \mathbb{Z} \to \mathbb{N} \cup \{0\} : x \mapsto |x|$. This is onto but not one-to-one.

EXAMPLE 2.4.8. Consider the function $f : \mathbb{R} \to \mathbb{R} : x \mapsto x/2$. This is one-to-one and onto. Also, it is easy to see that $f^{-1} : \mathbb{R} \to \mathbb{R} : x \mapsto 2x$.

EXAMPLE 2.4.9. Find whether the following yield functions from $\mathbb{N}$ to $\mathbb{N}$, and if so, whether they are one-to-one, onto or both. Find also the inverse function if the function is one-to-one and onto:

(1) $y = 2x + 3$;
(2) $y = 2x - 3$;
(3) $y = x^2$;
(4) $y = x + 1$ if x is odd, $y = x - 1$ if x is even.

Suppose that A, B and C are sets and that $f : A \to B$ and $g : B \to C$ are functions. We define the composition function $g \circ f : A \to C$ by writing $(g \circ f)(x) = g(f(x))$ for every $x \in A$.

ASSOCIATIVE LAW. *Suppose that A, B, C and D are sets, and that $f : A \to B$, $g : B \to C$ and $h : C \to D$ are functions. Then $h \circ (g \circ f) = (h \circ g) \circ f$.*

PROBLEMS FOR CHAPTER 2

1. The power set $\mathcal{P}(A)$ of a set A is the set of all subsets of A. Suppose that $A = \{1, 2, 3, 4, 5\}$.
   a) How many elements are there in $\mathcal{P}(A)$?
   b) How many elements are there in $\mathcal{P}(A \times \mathcal{P}(A)) \cup A$?
   c) How many elements are there in $\mathcal{P}(A \times \mathcal{P}(A)) \cap A$?

2. For each of the following relations $\mathcal{R}$ on $\mathbb{Z}$, determine whether the relation is reflexive, symmetric or transitive, and specify the equivalence classses if $\mathcal{R}$ is an equivalence relation on $\mathbb{Z}$:
   a) $(a, b) \in \mathcal{R}$ if a divides b
   b) $(a, b) \in \mathcal{R}$ if $a + b$ is even
   c) $(a, b) \in \mathcal{R}$ if $a + b$ is odd
   d) $(a, b) \in \mathcal{R}$ if $a \leq b$
   e) $(a, b) \in \mathcal{R}$ if $a^2 = b^2$
   f) $(a, b) \in \mathcal{R}$ if $a < b$

3. For each of the following relations $\mathcal{R}$ on $\mathbb{N}$, determine whether the relation is reflexive, symmetric or transitive, and specify the equivalence classses if $\mathcal{R}$ is an equivalence relation on $\mathbb{N}$:
   a) $(a, b) \in \mathcal{R}$ if $a < 3b$
   b) $(a, b) \in \mathcal{R}$ if $3a \leq 2b$
   c) $(a, b) \in \mathcal{R}$ if $a - b = 0$
   d) $(a, b) \in \mathcal{R}$ if 7 divides $3a + 4b$

4. Consider the set $A = \{1, 2, 3, 4, 6, 9\}$. Define a relation $\mathcal{R}$ on A by writing $(x, y) \in \mathcal{R}$ if and only if $x - y$ is a multiple of 3.
   a) Describe $\mathcal{R}$ as a subset of $A \times A$.
   b) Show that $\mathcal{R}$ is an equivalence relation on A.
   c) What are the equivalence classes of $\mathcal{R}$?

5. Let $A = \{1, 2, 4, 5, 7, 11, 13\}$. Define a relation $\mathcal{R}$ on A by writing $(x, y) \in \mathcal{R}$ if and only if $x - y$ is a multiple of 3.
   a) Show that $\mathcal{R}$ is an equivalence relation on A.
   b) How many equivalence classes of $\mathcal{R}$ are there?

6. Define a relation $\mathcal{R}$ on $\mathbb{Z}$ by writing $(x, y) \in \mathcal{R}$ if and only if $x - y$ is a multiple of 2 as well as a multiple of 3.
   a) Show that $\mathcal{R}$ is an equivalence relation on $\mathbb{Z}$.
   b) How many equivalence classes of $\mathcal{R}$ are there?

7. Define a relation $\mathcal{R}$ on $\mathbb{N}$ by writing $(x, y) \in \mathcal{R}$ if and only if $x - y$ is a multiple of 2 or a multiple of 3.
   a) Is $\mathcal{R}$ reflexive? Is $\mathcal{R}$ symmetric? Is $\mathcal{R}$ transitive?
   b) Find a subset A of $\mathbb{N}$ such that a relation $\mathcal{R}$ defined in a similar way on A is an equivalence relation.

8. Let $A = \{1, 2\}$ and $B = \{a, b, c\}$. For each of the following cases, decide whether the set represents the graph of a function $f : A \to B$; if so, write down $f(1)$ and $f(2)$, and determine whether f is one-to-one and whether f is onto:
   a) $\{(1, a), (2, b)\}$ b) $\{(1, b), (2, b)\}$ c) $\{(1, a), (1, b), (2, c)\}$

9. Let f, g and h be functions from $\mathbb{N}$ to $\mathbb{N}$ defined by
$$f(x) = \begin{cases} 1 & (x > 100), \\ 2 & (x \leq 100), \end{cases}$$
   $g(x) = x^2 + 1$ and $h(x) = 2x + 1$ for every $x \in \mathbb{N}$.
   a) Determine whether each function is one-to-one or onto.
   b) Find $h \circ (g \circ f)$ and $(h \circ g) \circ f$, and verify the Associative law for composition of functions.

10. Consider the function $f : \mathbb{N} \to \mathbb{N}$, given by $f(x) = x + 1$ for every $x \in \mathbb{N}$.
   a) What is the domain of this function?
   b) What is the range of this function?
   c) Is the function one-to-one?
   d) Is the function onto?

11. Let $f : A \to B$ and $g : B \to C$ be functions. Prove each of the following:
   a) If f and g are one-to-one, then $g \circ f$ is one-to-one.
   b) If $g \circ f$ is one-to-one, then f is one-to-one.
   c) If f is onto and $g \circ f$ is one-to-one, then g is one-to-one.
   d) If f and g are onto, then $g \circ f$ is onto.
   e) If $g \circ f$ is onto, then g is onto.
   f) If $g \circ f$ is onto and g is one-to-one, then f is onto.

12. a) Give an example of functions $f : A \to B$ and $g : B \to C$ such that $g \circ f$ is one-to-one, but g is not.
   b) Give an example of functions $f : A \to B$ and $g : B \to C$ such that $g \circ f$ is onto, but f is not.

13. Suppose that $f : A \to B$, $g : B \to A$ and $h : A \times B \to C$ are functions, and that the function $k : A \times B \to C$ is defined by $k(x, y) = h(g(y), f(x))$ for every $x \in A$ and $y \in B$.
   a) Show that if f, g and h are all one-to-one, then k is one-to-one.
   b) Show that if f, g and h are all onto, then k is onto.

14. Suppose that the set A contains 5 elements and the set B contains 2 elements.
   a) How many different functions $f : A \to B$ can one define?
   b) How many of the functions in part (a) are not onto?
   c) How many of the functions in part (a) are not one-to-one?

15. Suppose that the set A contains 2 elements and the set B contains 5 elements.
   a) How many of the functions $f : A \to B$ are not onto?
   b) How many of the functions $f : A \to B$ are not one-to-one?

16. Suppose that A, B, C and D are finite sets, and that $f : A \to B$, $g : B \to C$ and $h : C \to D$ are functions. Suppose further that the following four conditions are satisfied:
   - B, C and D have the same number of elements.
   - $f : A \to B$ is one-to-one and onto.
   - $g : B \to C$ is onto.
   - $h : C \to D$ is one-to-one.

   Prove that the composition function $h \circ (g \circ f) : A \to D$ is one-to-one and onto.

17. Let $A = \{1, 2\}$ and $B = \{2, 3, 4, 5\}$. Write down the number of elements in each of the following sets:
   a) $A \times A$
   b) the set of functions from A to B
   c) the set of one-to-one functions from A to B
   d) the set of onto functions from A to B
   e) the set of relations on B
   f) the set of equivalence relations on B for which there are exactly two equivalence classes
   g) the set of all equivalence relations on B
   h) the set of one-to-one functions from B to A
   i) the set of onto functions from B to A
   j) the set of one-to-one and onto functions from B to B

18. Define a relation $\mathcal{R}$ on $\mathbb{N} \times \mathbb{N}$ by $(a, b)\mathcal{R}(c, d)$ if and only if $a + b = c + d$.
   a) Prove that $\mathcal{R}$ is an equivalence relation on $\mathbb{N} \times \mathbb{N}$.
   b) Let S denote the set of equivalence classes of $\mathcal{R}$. Show that there is a one-to-one and onto function from S to $\mathbb{N}$.

19. Suppose that $\mathcal{R}$ is a relation defined on $\mathbb{N}$ by $(a, b) \in \mathcal{R}$ if and only if $[4/a] = [4/b]$. Here for every $x \in \mathbb{R}$, $[x]$ denotes the integer n satisfying $n \leq x < n + 1$.
   a) Show that $\mathcal{R}$ is an equivalence relation on $\mathbb{N}$.
   b) Let $\mathcal{S}$ denote the set of all equivalence classes of $\mathcal{R}$. Show that there is a one-to-one and onto function from $\mathcal{S}$ to $\{1, 2, 3, 4\}$.

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