The Sigma notation

This is Greek capital letter '**sigma**' :

$$\sum$$

as opposed to small letter s which you normally encounter in Statistics, where it usually stands for standard deviation.

$\sum$ is used to indicate the **summation** of a set or sequence of numbers or variables. In other words, the sigma notation can be expanded as a **series**.

You will most probably come across the sigma notation in the following form :

$$\sum_{r=a}^{b} f(r)$$

Let us first start by explaining the different notations above.

1) The variable r is known as the **index**.
2) The constants a and b are known as the **lower** and **upper limits** of summation respectively.
3) $f(r)$ is known as the **general term** of the series.

Remarks :

1) Other letters can be used for the index.
2) The constants a and b **must** be **integers**. This is because the index is an integer variable which starts at a. Its value is then incremented by steps of 1 until it finally reaches b.
3) The general term needs not always be in terms of r; it can be **independent** of r, in which case, it is treated as a **constant**.

The expansion and simplification of a given expression in terms of sigma is quite straightforward. Each time the index assumes a value between a and b inclusive, that value is substituted in the general term, in the place of the index, and one term of the series is obtained. The final value, or expression, depending on whether the limits are numbers or unknowns, is just the addition of all the terms of the series.

Here are some examples to illustrate how the sigma notation is simplified.

EXAMPLE 1

Evaluate $\sum_{k=2}^{6}(2k+1)$.

Solution

The index k is going to assume the values 2, 3, 4, 5 and 6 in turn. Thus,

$$\begin{aligned}\sum_{k=2}^{6}(2k+1) &= (2.2+1)+(2.3+1)+(2.4+1)+(2.5+1)+(2.6+1)\\ &= (4+1)+(6+1)+(8+1)+(10+1)+(12+1)\\ &= 5+7+9+11+13\\ &= 45.\end{aligned}$$

Note

Whenever the general term consists of a multiple of the index ($\pm$ a constant), the series thus formed is an **arithmetic series** with a common difference equal to the coefficient of the index (2, in the above case).

EXAMPLE 2

Simplify $\sum_{r=0}^{5}2^r$.

Solution

Again, by substitution, the index r will take on the values 0 to 5 inclusive and

$$\begin{aligned}\sum_{r=0}^{5}2^r &= 2^0+2^1+2^2+2^3+2^4+2^5\\ &= 1+2+4+8+16+32\\ &= 63.\end{aligned}$$

Note

Whenever the general term consists of a number whose exponent is the index, the series thus formed is a **geometric series** with a common ratio equal to that number(2, in the above case).

EXAMPLE 3

Expand $\sum_{i=1}^{4} x_i$.

Solution

$$\sum_{i=1}^{4} x_i \;=\; x_1 + x_2 + x_3 + x_4 .$$

Note

The above example is just a general illustration of the summation notation. The expression cannot be simplified any further.

We can also work in the reverse order, that is, express a given expansion in summation notation.

EXAMPLE 4

Express $1+4+7+10+13$ in summation form.

Solution

We notice that the above is an arithmetic series with common difference 3. Using the same reasoning as that in Example 1, the general term must be of the form $3r \pm c$, where *r* is the index and *c*, a constant. However, the choice of the lower limit of summation determines the value of *c* and vice versa. This leaves us with an infinite number of possible answers consisting of different pairs of limits and general terms.

One such possibility would be to choose the lower limit as 1. The value of c can be found by equating $3r + c$ to 1. With r =1, this gives $c = -2$. The upper limit of summation can be obtained by solving the equation $3r + c$ = 13 or by simply counting the number of terms ! The answer would be

$$\sum_{r=1}^{5} (3r - 2)$$

We might as well choose the lower limit as 0, in which case, it is easy to show that *c* is 1 and that the corresponding answer is

$$\sum_{r=0}^{4} (3r + 1)$$

EXAMPLE 5

Express $3+6+12+24+48+96$ in summation form.

Solution

This time, we notice that the above series is a geometric series with common ratio 2. Using the reasoning in Example 2, we know that the general term must contain the term 2^r. In fact, the above series can be expressed as $3\times(1+2+4+8+16+32)$ and thus

$$3+6+12+24+48+96 = 3\times\sum_{r=0}^{5} 2^r \ ,$$

which is the required answer.

EXAMPLE 6

Express $1-2+4-8+16-32$ in summation form.

Solution

This is an example of an **alternating** series. It is a geometric series with common ratio -2. With the usual technique, we can write the answer as

$$1-2+4-8+16-32 = \sum_{r=0}^{5}(-2)^r \quad \text{or} \quad \sum_{r=0}^{5}(-1)^r(2)^r$$

Note

The series $-1+2-4+8-16+32$ would be written as $\sum_{r=0}^{5}(-1)^{r+1}(2)^r$. The $+1$ in the exponent of -1 has the beautiful effect of shifting the alternating plus and minus signs by one place ! We deduce that the presence of the power series of -1 is essential whenever an alternating series is expressed in summation notation.

LAWS OF SUMMATION

1. $\sum_{r=1}^{n} c = nc$, where c is a constant.

Remark

Whenever the general term is a constant, that is, independent of the index, the latter acts as a counter. A more generalised result would be

$$\sum_{r=a}^{b} c = (b-a+1)\,c,$$

that is, the product of the number of terms in the summation and the constant.

Example

$\sum_{r=4}^{7} 3 = (7-3)\times 3 = 12.$

2. $\sum_{r=1}^{n} b\,f(r) \; = \; b\sum_{r=1}^{n} f(r)$

Any constant multiple of a function can be taken outside the summation sign. This can be easily proven by the following example.

Example

$$\sum_{r=1}^{3} 5.3^{r} = 5.3^{1} + 5.3^{2} + 5.3^{3} = 5\times(3^{1} + 3^{2} + 3^{3}) = 5\times\sum_{r=1}^{3} 3^{r}$$

Remark

The above case has its corresponding law in integration.

3. $\sum_{r=1}^{n} [\,f(r) \pm g(r)] = \sum_{r=1}^{n} f(r) \pm \sum_{r=1}^{n} g(r)$

Summation is **distributive** over the addition operation. This result is similar to that in integration and can be easily proven as an exercise for the student.

Exercise

Try proving that $\sum_{r=1}^{4} (r + r^{2}) = \sum_{r=1}^{4} r + \sum_{r=1}^{4} r^{2}$

Important note

Summation is **not** distributive over the multiplication operation. Thus,

1) $\sum_{r=1}^{n} f(r).g(r) \neq \sum_{r=1}^{n} f(r) \sum_{r=1}^{n} g(r)$

2) $\sum_{r=1}^{n} \frac{f(r)}{g(r)} \neq \frac{\sum_{r=1}^{n} f(r)}{\sum_{r=1}^{n} g(r)}$

FAQ

What happens when the number of terms in the summation is very large? Well, obviously, it will not be practical to expand the series and sum all the terms. In that case, we will make use of **formulae**. These formulae help us shorten any calculation and, in addition, their derivations are beyond the scope of this course! We don't even need to learn them by heart because they are given in the formula sheet in the examinations!
Therefore, we must just know how to apply them efficiently.

FORMULAE

1. $\sum_{r=1}^{n} r = \frac{n(n+1)}{2}$

Remark

This is the sum of the first n positive integers. It is an arithmetic series with first term 1 and common difference 1.

2. $\sum_{r=1}^{n} r^2 = \frac{n(n+1)(2n+1)}{6}$

3. $\sum_{r=1}^{n} r^3 = \left[\frac{n(n+1)}{2}\right]^2$

Note

These formulae can only be directly applied if the lower limit of summation is 1 in any given problem.

We shall now illustrate the application of the laws and formulae of summation to the different types of examination problems which are very often encountered by Higher School Certificate students.

EXAMPLE 7

Express $\sum_{r=0}^{n}(2n+1-2r)$ in terms of n.

Solution

Recall : Any term which is independent of the index is treated as a constant. Therefore,

$$\begin{aligned}\sum_{r=0}^{n}(2n+1-2r) &= \sum_{r=0}^{n}(2n+1)-\sum_{r=0}^{n}2r = \sum_{r=0}^{n}(2n+1)-2\sum_{r=0}^{n}r \\ &= (n+1)(2n+1)-\frac{2n(n+1)}{2}\end{aligned}$$

since there are $(n+1)$ terms from 0 to n. Hence,

$$\sum_{r=0}^{n}(2n+1-2r) = (n+1)[(2n+1)-n]=(n+1)(n+1)=(n+1)^2.$$

Note

$$\sum_{r=0}^{n}r = 0+1+2+...+n = 1+2+3+...+n = \sum_{r=1}^{n}r = \frac{n(n+1)}{2}$$

EXAMPLE 8

Prove that $\sum_{r=1}^{n}(3r^2-3r+1)=n^3$.

Solution

Using the laws of summation, we have

$$\sum_{r=1}^{n}(3r^2-3r+1)=3\sum_{r=1}^{n}r^2-3\sum_{r=1}^{n}r+\sum_{r=1}^{n}1$$

Now, applying the formulae for summation,

$$\sum_{r=1}^{n}(3r^2-3r+1)=\frac{3n(n+1)(2n+1)}{6}-\frac{3n(n+1)}{2}+n$$

By simplifying further, we have

$$\sum_{r=1}^{n}(3r^2-3r+1)=\frac{n}{2}\left[(n+1)(2n+1)-3(n+1)+2\right]$$

$$\Rightarrow \sum_{r=1}^{n}(3r^2-3r+1)=\frac{n}{2}\left[2n^2+3n+1-3n-3+2\right]$$

$$=\frac{n}{2}\times 2n^2=n^3.$$

EXAMPLE 9

Express $\sum_{r=n+1}^{2n}(5r-3)$ in terms of n.

Solution

Note
The lower limit of summation is **not** 1! Hence, this does not allow direct application of any formula.

We break down the given summation into two other summations which both have a lower limit of 1. It should be clear that the sum from the $(n + 1)^{\text{th}}$ term to the $(2n)^{\text{th}}$ term is the same as the difference between the sum of the first $2n$ terms and the sum of the first n terms.

This is done as follows :

$$\sum_{r=n+1}^{2n}(5r-3)=\sum_{r=1}^{2n}(5r-3)-\sum_{r=1}^{n}(5r-3)$$

Now we have two summations which can be easily calculated because their lower limits are both 1.

$$\sum_{r=1}^{2n}(5r-3)-\sum_{r=1}^{n}(5r-3)=5\sum_{r=1}^{2n}r-\sum_{r=1}^{2n}3-5\sum_{r=1}^{n}r+\sum_{r=1}^{n}3$$

$$=\frac{5\times 2n(2n+1)}{2}-(3\times 2n)-\frac{5n(n+1)}{2}+3n$$

$$=\frac{n}{2}\left[10(2n+1)-12-5(n+1)+6\right]$$

$$=\frac{n}{2}\left[20n+10-12-5n-5+6\right]$$

$$=\frac{n}{2}(15n-1).$$

EXAMPLE 10

Given that $\sum_{r=1}^{n}(ar+b) \equiv n^2$, find the constants a and b.

Solution

$$\text{L.H.S} = \sum_{r=1}^{n}(ar+b) = a\sum_{r=1}^{n}r + \sum_{r=1}^{n}b = \frac{an(n+1)}{2} + nb$$

$$= \frac{an(n+1)+2nb}{2} = \frac{an^2+an+2nb}{2} = \left(\frac{a}{2}\right)n^2 + \left(\frac{a+2b}{2}\right)n$$

Equating coefficients of n and n^2 on the L.H.S and R.H.S, we have

$$\frac{a+2b}{2} = 0 \quad \text{and} \quad \frac{a}{2} = 1.$$

Thus, $a = 2$ and $a = -2b \Rightarrow b = -1$.

EXAMPLE 11

Evaluate $\sum_{r=1}^{\infty}\frac{1}{10^{3r}}$. Hence, or otherwise, write the recurring decimal $0.\dot{2}3\dot{7}$ as a fraction in its simplest form.

Solution

The expansion of the summation

$$\sum_{r=1}^{\infty}\frac{1}{10^{3r}} = \frac{1}{10^3} + \frac{1}{10^6} + \frac{1}{10^9} + \ldots\ldots$$

gives us a geometric series with first term and common ratio $\frac{1}{10^3}$. Since the common ratio is a fraction, a sum to infinity does exist. Therefore,

$$\sum_{r=1}^{\infty}\frac{1}{10^{3r}} = \frac{\frac{1}{10^3}}{1-\frac{1}{10^3}} = \frac{1}{999}.$$

Now, $0.\dot{2}3\dot{7} = 0.237237237237\ldots..$

$$= 0.237 + 0.000237 + 0.000000237 + \ldots..$$

$$= 237\times\left(\frac{1}{10^3} + \frac{1}{10^6} + \frac{1}{10^9} + \ldots\ldots\right)$$

$$= 237\times\frac{1}{999} = \frac{237}{999} = \frac{79}{333}.$$

EXAMPLE 12

Express $\dfrac{2}{4x^2-1}$ in partial fractions.

Hence, or otherwise, show that $\displaystyle\sum_{r=1}^{n}\frac{2}{4r^2-1}=\frac{2n}{2n+1}$.

Solution

Since $\dfrac{2}{4x^2-1}\equiv\dfrac{2}{(2x-1)(2x+1)}$, let $\dfrac{2}{4x^2-1}\equiv\dfrac{A}{(2x-1)}+\dfrac{B}{(2x+1)}$ where A and B are two arbitrary constants to be found. Multiplying L.H.S and R.H.S by $(2x-1)(2x+1)$, we have the identity

$$2\equiv A(2x+1)+B(2x-1)$$

When $x=\dfrac{1}{2}$, $2\equiv A(2)+B(0)\Rightarrow A=1$.

When $x=-\dfrac{1}{2}$, $2\equiv A(0)+B(-2)\Rightarrow B=-1$.

Therefore, $\dfrac{2}{4x^2-1}\equiv\dfrac{1}{(2x-1)}-\dfrac{1}{(2x+1)}$.

In this case, we cannot use any of the three given formulae because the index is found in the denominator of the general term. Using the above result, we can write

$$\sum_{r=1}^{n}\frac{2}{4r^2-1}=\sum_{r=1}^{n}\left(\frac{1}{2r-1}-\frac{1}{2r+1}\right)=\sum_{r=1}^{n}\frac{1}{2r-1}-\sum_{r=1}^{n}\frac{1}{2r+1}.$$

Expanding each summation (just to have an idea of how the terms look like!), we have

$$\sum_{r=1}^{n}\frac{2}{4r^2-1}=\left(1-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{5}\right)+\left(\frac{1}{5}-\frac{1}{7}\right)+........+\left(\frac{1}{2n-3}-\frac{1}{2n-1}\right)+\left(\frac{1}{2n-1}-\frac{1}{2n+1}\right).$$

We can see that by adding the terms of the two summations, most of them get cancelled! In fact, we are left with only $1-\dfrac{1}{2n+1}=\dfrac{2n}{2n+1}$.

Remark

AVOID GUESSING! On the contrary, when in doubt, proceed by trial and error.