TRAPEZOIDAL METHOD
ERROR FORMULA

<u>Theorem</u>. Let $f(x)$ have two continuous derivatives on the interval $a \leq x \leq b$. Then

$$E_n^T(f) \equiv \int_a^b f(x)\,dx - T_n(f) = -\frac{h^2\,(b-a)}{12} f''\,(c_n)$$

for some c_n in the interval $[a, b]$.

Later I will say something about the proof of this result, as it leads to some other desirable formulas for the error.

The above formula says that the error decreases in a manner that is roughly proportional to h^2. Thus doubling n (and halving h) should cause the error to decrease by a factor of approximately 4. This is what we observed with a past example from the last lecture.

EXAMPLE

Consider evaluating

$$I = \int_0^2 \frac{dx}{1+x^2}$$

using the trapezoidal method $T_n(f)$. How large should n be chosen in order to ensure that

$$\left|E_n^T(f)\right| \leq 5 \times 10^{-6}$$

We begin by calculating the derivatives:

$$f'(x) = \frac{-2x}{\left(1+x^2\right)^2}, \qquad f''(x) = \frac{-2+6x^2}{\left(1+x^2\right)^3}$$

From a graph of $f''(x)$,

$$\max_{0\leq x\leq 2} \left|f''(x)\right| = 2$$

Recall that $nh = b - a = 2$. Therefore,

$$\begin{aligned} E_n^T(f) &= -\frac{h^2(b-a)}{12} f''(c_n) \\ \left|E_n^T(f)\right| &\leq \frac{h^2 2}{12} \cdot 2 = \frac{h^2}{3} \end{aligned}$$

$$
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\end{aligned}
$$

We bound $|f''(c_n)|$ since we do not know c_n, and therefore we must assume the worst possible case, that which makes the error formula largest. That is what has been done above.

When do we have

$$
\left| E_n^T(f) \right| \le 5 \times 10^{-6} \qquad (*)
$$

To ensure this, we choose h so small that

$$
\frac{h^2}{3} \le 5 \times 10^{-6}
$$

This is equivalent to choosing h and n to satisfy

$$
\begin{aligned}
h &\le .003873 \\
n = \frac{2}{h} &\ge 516.4
\end{aligned}
$$

Thus $n \ge 517$ will imply $(*)$.

DERIVING THE ERROR FORMULA

There are two stages in deriving the error.
First we obtain the error formula for the case of a single subinterval ($n = 1$), and second we use this to obtain the general error formula given earlier.

For the trapezoidal method with only a single subinterval, we have

$$\int_{\alpha}^{\alpha+h} f(x)\,dx - \frac{h}{2}\left[f(\alpha) + f(\alpha+h)\right] = -\frac{h^3}{12} f''(c)$$

for some c in the interval $[\alpha, \alpha + h]$.

We discuss later the derivation of this error formula. Note the error in the text in formula (7.26) on page 174, where the negative sign is missing.

Recall that the general trapezoidal rule $T_n(f)$ was obtained by applying the simple trapezoidal rule to a subdivision of the original interval of integration. Recall defining and writing

$$h = \frac{b-a}{n}, \qquad x_j = a + j\,h, \quad j = 0, 1, ..., n$$

$$\begin{aligned} I &= \int_{x_0}^{x_n} f(x)\,dx \\ &= \int_{x_0}^{x_1} f(x)\,dx + \int_{x_1}^{x_2} f(x)\,dx + \cdots + \int_{x_{n-1}}^{x_n} f(x)\,dx \end{aligned}$$

$$\begin{aligned} I \approx \tfrac{h}{2}\,[f(x_0) + f(x_1)] + \tfrac{h}{2}\,[f(x_1) + f(x_2)] \\ + \cdots + \tfrac{h}{2}\,[f(x_{n-2}) + f(x_{n-1})] + \tfrac{h}{2}\,[f(x_{n-1}) + f(x_n)] \end{aligned}$$

Then the error

$$E_n^T(f) \equiv \int_a^b f(x)\,dx - T_n(f)$$

can be analyzed by adding together the errors over the subintervals $[x_0, x_1]$, $[x_1, x_2]$, ..., $[x_{n-1}, x_n]$. Recall

$$\int_\alpha^{\alpha+h} f(x)\,dx - \frac{h}{2}\left[f(\alpha) + f(\alpha + h)\right] = -\frac{h^3}{12} f''(c)$$

Then on $[x_{j-1}, x_j]$,

$$\int_{x_{j-1}}^{x_j} f(x)\,dx - \frac{h}{2}\Big[f(x_{j-1}) + f(x_j)\Big] = -\frac{h^3}{12} f''(\gamma_j)$$

with $x_{j-1} \leq \gamma_j \leq x_j$, but otherwise γ_j unknown. Then combining these errors, we obtain

$$E_n^T(f) = -\frac{h^3}{12} f''(\gamma_1) - \cdots - \frac{h^3}{12} f''(\gamma_n)$$

This formula can be further simplified, and we will do so in more than one way.

Rewrite this error as

$$E_n^T(f) = -\frac{h^3 n}{12}\left[\frac{f''(\gamma_1) + \cdots + f''(\gamma_n)}{n}\right]$$

Denote the quantity inside the brackets by ζ_n. This number satisfies

$$\min_{a \le x \le b} f''(x) \le \zeta_n \le \max_{a \le x \le b} f''(x)$$

Since $f''(x)$ is a continuous function (by original assumption), we have that there must be some number c_n in $[a, b]$ for which

$$f''(c_n) = \zeta_n$$

Recall also that $hn = b - a$. Then

$$\begin{aligned} E_n^T(f) &= -\frac{h^3 n}{12}\left[\frac{f''(\gamma_1) + \cdots + f''(\gamma_n)}{n}\right] \\ &= -\frac{h^2 (b-a)}{12} f''(c_n) \end{aligned}$$

This is the error formula given on the first slide.

AN ERROR ESTIMATE

We now obtain a way to estimate the error $E_n^T(f)$. Return to the formula

$$E_n^T(f) = -\frac{h^3}{12}f''(\gamma_1) - \cdots - \frac{h^3}{12}f''(\gamma_n)$$

and rewrite it as

$$E_n^T(f) = -\frac{h^2}{12}\left[f''(\gamma_1)h + \cdots + f''(\gamma_n)h\right]$$

The quantity

$$f''(\gamma_1)h + \cdots + f''(\gamma_n)h$$

is a <u>Riemann sum</u> for the integral

$$\int_a^b f''(x)\,dx = f'(b) - f'(a)$$

By this we mean

$$\lim_{n\to\infty}\left[f''(\gamma_1)h + \cdots + f''(\gamma_n)h\right] = \int_a^b f''(x)\,dx$$

Thus

$$f''(\gamma_1)h + \cdots + f''(\gamma_n)h \approx f'(b) - f'(a)$$

for larger values of n. Combining this with the earlier error formula

$$E_n^T(f) = -\frac{h^2}{12}\Big[f''(\gamma_1)h + \cdots + f''(\gamma_n)h\Big]$$

we have

$$E_n^T(f) \approx -\frac{h^2}{12}\Big[f'(b) - f'(a)\Big] \equiv \widetilde{E}_n^T(f)$$

This is a computable estimate of the error in the numerical integration. It is called an 'asymptotic error estimate'.

EXAMPLE

Consider evaluating

$$I(f) = \int_0^{\pi} e^x \cos x \, dx = -\frac{e^{\pi}+1}{2} \doteq -12.070346$$

In this case,

$$\begin{aligned} f'(x) &= e^x [\cos x - \sin x] \\ f''(x) &= -2e^x \sin x \\ \max_{0 \le x \le \pi} |f''(x)| &= |f''(.75\pi)| = 14.921 \end{aligned}$$

Then

$$\begin{aligned} E_n^T(f) &= -\frac{h^2 (b-a)}{12} f''(c_n) \\ \left|E_n^T(f)\right| &\le \frac{h^2 \pi}{12} \cdot 14.921 = 3.906 h^2 \end{aligned}$$

Also

$$\begin{aligned} \widetilde{E}_n^T(f) &= -\frac{h^2}{12} \left[f'(\pi) - f'(0)\right] \\ &= \frac{h^2}{12} [e^{\pi} + 1] \doteq 2.012 h^2 \end{aligned}$$

In looking at the table (on a separate page) for evaluating the integral I by the trapezoidal, we see that the error $E_n^T(f)$ and the error estimate $\widetilde{E}_n^T(f)$ are quite close. Therefore

$$\begin{aligned} I(f) - T_n(f) &\approx \frac{h^2}{12}\left[e^\pi + 1\right] \\ I(f) &\approx T_n(f) + \frac{h^2}{12}\left[e^\pi + 1\right] \end{aligned}$$

This last formula is called the 'corrected trapezoidal rule', and it is illustrated in the second table (on the separate page). We see it gives a much smaller error for essentially the same amount of work; and it converges much more rapidly.

In general,

$$\begin{aligned} I(f) - T_n(f) &\approx -\frac{h^2}{12}\left[f'(b) - f'(a)\right] \\ I(f) &\approx T_n(f) - \frac{h^2}{12}\left[f'(b) - f'(a)\right] \end{aligned}$$

This is the 'corrected trapezoidal rule'. It is easy to obtain from the trapezoidal rule, and in most cases, it converges more rapidly than the trapezoidal rule.