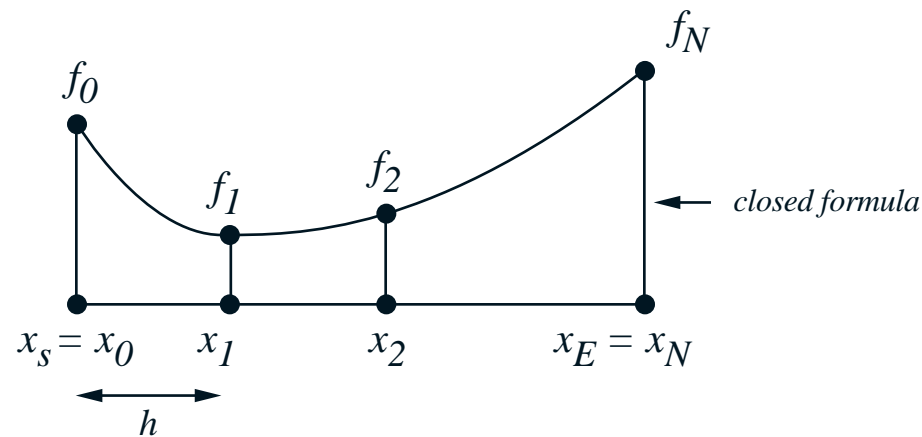


LECTURE 19

GAUSS QUADRATURE

- In general for Newton-Cotes (equispaced interpolation points/ data points/ integration points/ nodes).

$$\int_{x_S}^{x_E} f(x) dx = h[w'_0 f_0 + w'_1 f_1 + \dots + w'_N f_N] + E$$

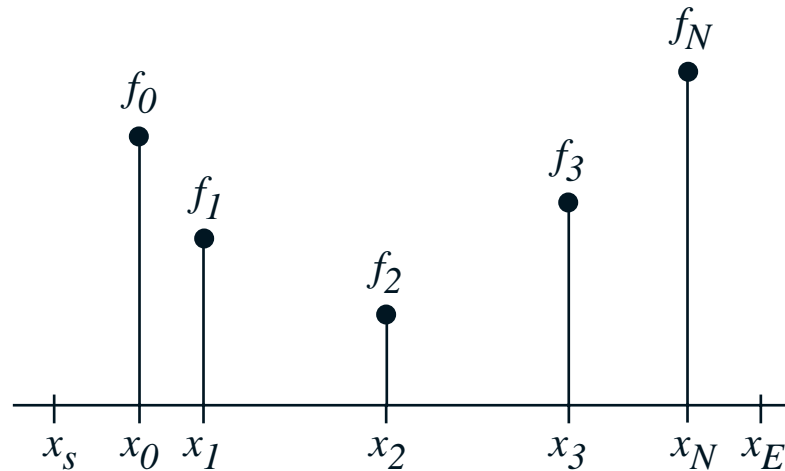


- Note that for Newton-Cotes formulae only the weighting coefficients w_i were unknown and the x_i were fixed

- However the number of and placement of the integration points influences the accuracy of the Newton-Cotes formulae:
 - N even $\rightarrow N^{th}$ degree interpolation function exactly integrates an $N + 1^{th}$ degree polynomial \rightarrow This is due to the placement of one of the data points.
 - N odd $\rightarrow N^{th}$ degree interpolation function exactly integrates an N^{th} degree polynomial.
- *Concept: Let's allow the placement of the integration points to vary such that we further increase the degree of the polynomial we can integrate exactly for a given number of integration points.*
- *In fact we can integrate an $2N + 1$ degree polynomial exactly with only $N + 1$ integration points*

- Assume that for Gauss Quadrature the form of the integration rule is

$$\int_{x_S}^{x_E} f(x) dx = [w_0 f_0 + w_1 f_1 + \dots + w_N f_N] + E$$



- In *deriving* (not applying) these integration formulae
 - Location of the integration points, x_i $i = 0, N$ are unknown
 - Integration formulae weights, w_i $i = 0, N$ are unknown
- $2(N + 1)$ unknowns \rightarrow we will be able to exactly integrate any $2N + 1$ degree polynomial!

Derivation of Gauss Quadrature by Integrating Exact Polynomials and Matching

Derive 1 point Gauss-Quadrature

- 2 unknowns w_o , x_o which will exactly integrate any linear function
- Let the *general* polynomial be

$$f(x) = Ax + B$$

where the coefficients A , B can equal any value

- Also consider the integration interval to be $[-1, +1]$ such that $x_S = -1$ and $x_E = +1$ (no loss in generality since we can always transform coordinates).

$$\int_{-1}^{+1} f(x)dx = w_o f(x_o)$$

- Substituting in the form of $f(x)$

$$\int_{-1}^{+1} (Ax + B)dx = w_o(Ax_o + B) \Rightarrow$$

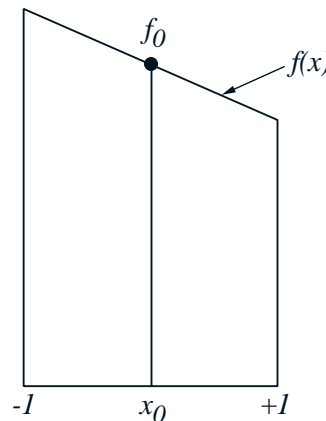
$$\left[A \frac{x^2}{2} + Bx \right]_{-1}^{+1} = w_o(Ax_o + B) \Rightarrow$$

$$A(0) + B(2) = A(x_o w_o) + B(w_o)$$

- In order for this to be true for any 1st degree polynomial (i.e. any A and B).

$$\begin{cases} 0 = x_o w_o \\ 2 = w_o \end{cases}$$

- Therefore $x_o = 0$, $w_o = 2$ for 1 point ($N = 1$) Gauss Quadrature.



- We can integrate exactly with only 1 point for a linear function while for Newton-Cotes we needed two points!

Derive a 2 point Gauss Quadrature Formula



- The general form of the integration formula is

$$I = w_0 f_0 + w_1 f_1$$

- w_0, x_0, w_1, x_1 are all unknowns
- 4 unknowns \Rightarrow we can fit a 3rd degree polynomial exactly

$$f(x) = Ax^3 + Bx^2 + Cx + D$$

- Substituting in for $f(x)$ into the general form of the integration rule

$$\int_{-1}^{+1} f(x) dx = w_0 f(x_0) + w_1 f(x_1)$$

\Rightarrow

$$\int_{-1}^{+1} [Ax^3 + Bx^2 + Cx + D]dx = w_o[Ax_o^3 + Bx_o^2 + Cx_o + D] + w_1[Ax_1^3 + Bx_1^2 + Cx_1 + D]$$

$$\Rightarrow$$

$$\left[\frac{Ax^4}{4} + \frac{Bx^3}{3} + \frac{Cx^2}{2} + Dx \right]_{-1}^{+1} = w_o(Ax_o^3 + Bx_o^2 + Cx_o + D) + w_1(Ax_1^3 + Bx_1^2 + Cx_1 + D)$$

$$\Rightarrow$$

$$A[w_o x_o^3 + w_1 x_1^3] + B\left[w_o x_o^2 + w_1 x_1^2 - \frac{2}{3}\right] + C[w_o x_o + w_1 x_1] + D[w_o + w_1 - 2] = 0$$

- In order for this to be true for **any** third degree polynomial (i.e. all arbitrary coefficients, A, B, C, D), we must have:

$$w_o x_o^3 + w_1 x_1^3 = 0$$

$$w_o x_o^2 + w_1 x_1^2 - \frac{2}{3} = 0$$

$$w_o x_o + w_1 x_1 = 0$$

$$w_o + w_1 - 2 = 0$$

- 4 nonlinear equations \rightarrow 4 unknowns

$$w_0 = 1 \quad \text{and} \quad w_1 = 1$$

$$x_0 = -\sqrt{\frac{1}{3}} \quad \text{and} \quad x_1 = +\sqrt{\frac{1}{3}}$$

- All polynomials of degree 3 or less will be *exactly* integrated with a Gauss-Legendre 2 point formula.

Gauss Legendre Formulae

$$I = \int_{-1}^{+1} f(x) dx = \sum_{i=0}^N w_i f_i + E$$

N	$N + 1$	$x_i,$ $i = 0, N$	w_i	Exact for polynomials of degree
0	1	0	2	1
1	2	$-\sqrt{\frac{1}{3}}, +\sqrt{\frac{1}{3}}$	1, 1	3
2	3	-0.774597, 0, +0.774597	0.5555, 0.8889, 0.5555	5
N	$N + 1$			$2N + 1$

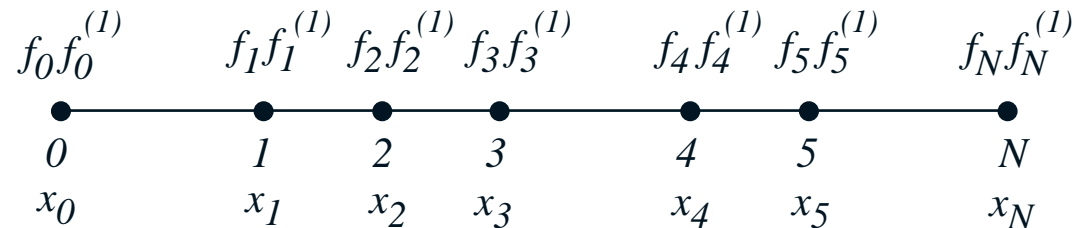
- Notes
 - $N + 1 =$ the number of integration points
 - Integration points are symmetrical on $[-1, +1]$
 - Formulae can be applied on any interval using a coordinate transformation
 - $N + 1$ integration points \rightarrow will integrate polynomials of up to degree $2N + 1$ exactly.
 - Recall that Newton Cotes $\rightarrow N + 1$ integration points only integrates an $N^{th}/N + 1^{th}$ degree polynomial exactly depending on N being odd or even.
 - For Gauss-Legendre integration, we allowed both weights and integration point locations to vary to match an integral exactly \Rightarrow more d.o.f. \Rightarrow allows you to match a higher degree polynomial!
 - An alternative way of looking at Gauss-Legendre integration formulae is that we use Hermite interpolation instead of Lagrange interpolation! (How can this be since Hermite interpolation involves derivatives \rightarrow let's examine this!)

Derivation of Gauss Quadrature by Integrating Hermite Interpolating Functions

Hermite interpolation formulae

- Hermite interpolation which *matches* the function and the first derivative at $N + 1$ interpolation points is expressed as:

$$g(x) = \sum_{i=0}^N \alpha_i(x) f_i + \sum_{i=0}^N \beta_i(x) f_i^{(1)}$$



- It can be shown that in general for non-equispaced points

$$\alpha_i(x) = t_i(x) l_{iN}(x) l_{iN}(x) \quad i = 0, N$$

$$\beta_i(x) = s_i(x) l_{iN}(x) l_{iN}(x) \quad i = 0, N$$

where

$$p_N(x) \equiv (x - x_0)(x - x_1) \cdots (x - x_N)$$

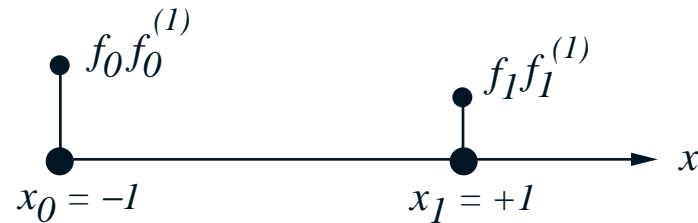
$$l_{iN}(x) \equiv \frac{p_N(x)}{(x - x_i) p_N^{(1)}(x_i)} \quad i = 0, N$$

$$t_i(x) \equiv 1 - (x - x_i)^2 l_{iN}^{(1)}(x_i)$$

$$s_i(x) \equiv (x - x_i)$$

Example of defining a cubic Hermite interpolating function

- Derive Hermite interpolating functions for 2 interpolation points located at -1 and $+1$ for the interval $[-1, +1]$.



$$N + 1 = 2 \text{ points} \Rightarrow N = 1$$

- Establish $p_N(x)$

$$p_1(x) = (x - x_0)(x - x_1) \Rightarrow$$

$$p_1^{(1)}(x) = (x - x_0) + (x - x_1)$$

- Establish $l_{iN}(x)$

$$l_{i1}(x) = \frac{p_1(x)}{(x - x_i)[p_1^{(1)}(x_i)]}$$

$$l_{i1}(x) = \frac{(x - x_o)(x - x_1)}{(x - x_i)[(x_i - x_o) + (x_i - x_1)]}$$

- Let $i = 0$

$$l_{o1}(x) = \frac{(x - x_o)(x - x_1)}{(x - x_o)[0 + (x_o - x_1)]} \Rightarrow$$

$$l_{o1}(x) = \frac{x - x_1}{x_o - x_1}$$

- Substitute in $x_o = -1$ and $x_1 = +1$

$$l_{o1}(x) = \frac{1}{2}(1 - x)$$

- Let $i = 1$

$$l_{11}(x) = \frac{(x - x_o)(x - x_1)}{(x - x_1)[(x_1 - x_o) + 0]}$$

- Substitute in values for x_o, x_1

$$l_{11}(x) = \frac{1}{2}(1 + x)$$

- Taking derivatives

$$l_{o1}^{(1)}(x) = -\frac{1}{2}$$

$$l_{11}^{(1)}(x) = +\frac{1}{2}$$

- Establish $t_i(x)$

$$t_o(x) = 1 - (x - x_o) 2 l_{o1}^{(1)}(x_o) \quad \Rightarrow$$

$$t_o(x) = 1 - (x + 1)(2) \left(-\frac{1}{2}\right) \quad \Rightarrow$$

$$t_o(x) = 2 + x$$

$$t_1(x) = 1 - (x - x_1) 2 l_{11}^{(1)}(x_1) \quad \Rightarrow$$

$$t_1(x) = 1 - (x - 1) \left(2 \cdot \frac{1}{2}\right) \quad \Rightarrow$$

$$t_1(x) = 2 - x$$

- Establish $s_i(x)$

$$s_o(x) = x + 1$$

$$s_1(x) = x - 1$$

- Establish $\alpha_i(x)$

$$\alpha_o(x) = t_o(x)l_{o1}(x)l_{o1}(x) \Rightarrow$$

$$\alpha_o(x) = (2+x)\frac{1}{2}(1-x)\frac{1}{2}(1-x) \Rightarrow$$

$$\alpha_o(x) = \frac{1}{4}(2 - 3x + x^3)$$

$$\alpha_1(x) = t_1(x)l_{11}(x)l_{11}(x) \Rightarrow$$

$$\alpha_1(x) = (2-x)\frac{1}{2}(1+x)\frac{1}{2}(1+x) \Rightarrow$$

$$\alpha_1(x) = \frac{1}{4}(2 + 3x - x^3)$$

- Establish $\beta_i(x)$

$$\beta_o(x) = s_o(x)l_{o1}(x)l_{o1}(x) \Rightarrow$$

$$\beta_o(x) = (x+1)\frac{1}{2}(1-x)\frac{1}{2}(1-x) \Rightarrow$$

$$\beta_o(x) = \frac{1}{4}(1-x-x^2+x^3)$$

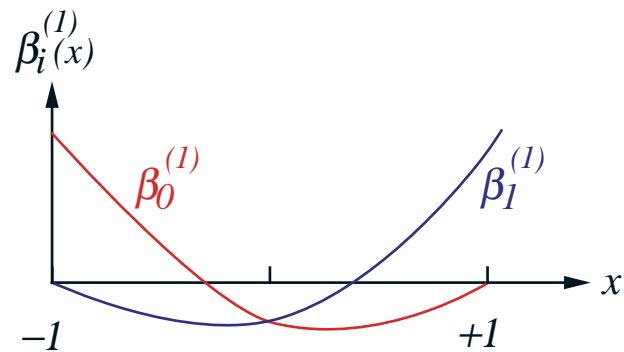
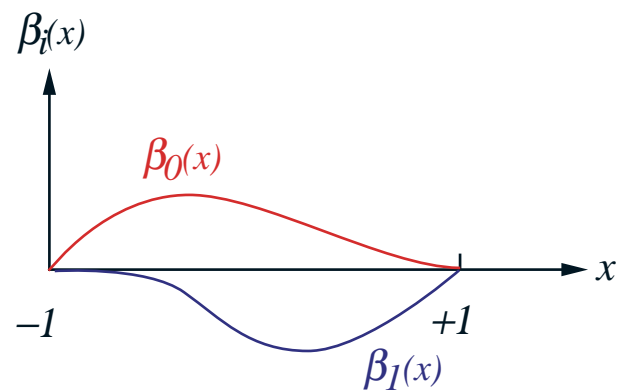
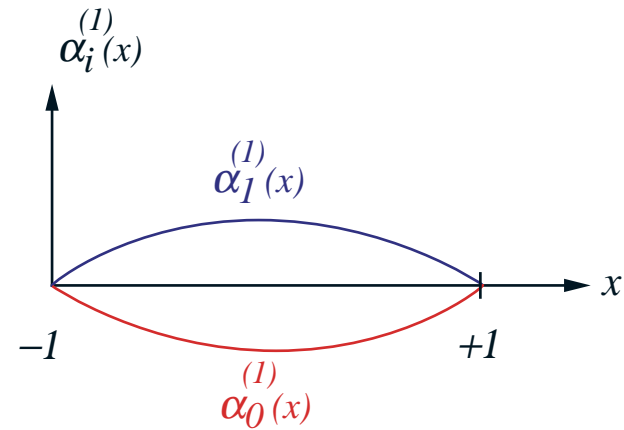
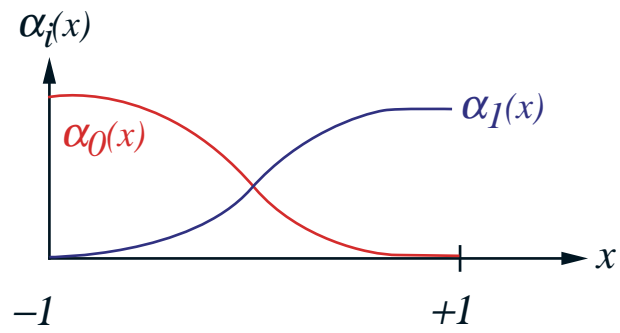
$$\beta_1(x) = s_1(x)l_{11}(x)l_{11}(x) \Rightarrow$$

$$\beta_1(x) = (x-1)\frac{1}{2}(1+x)\frac{1}{2}(1+x) \Rightarrow$$

$$\beta_1(x) = \frac{1}{4}(-1-x+x^2+x^3)$$

- In general

$$g(x) = \alpha_o(x) f_o + \alpha_1(x) f_1 + \beta_o(x) f_o^{(1)} + \beta_1(x) f_1^{(1)}$$



- These functions satisfy the constraints

$$\alpha_i(x_j) = \delta_{ij}$$

$$\beta_i(x_j) = 0$$

$$\alpha_i^{(1)}(x_j) = 0$$

$$\beta_i^{(1)}(x_j) = \delta_{ij}$$

Gauss-Legendre Quadrature by integrating Hermite interpolating polynomials

$$I = \int_{-1}^{+1} f(x) dx = \sum_{i=0}^N w_i f_i + E$$

- Notes
 - Use $[-1, +1]$ without loss of generality \Rightarrow we can always transform the interval.
 - Approximation for I is exact for $2N + 1$ degree polynomials
- We can derive all Gauss-Legendre quadrature formulae by approximating $f(x)$ with an $2N + 1^{th}$ degree Hermite interpolating function *using* N *specially selected* integration/interpolation points.

$$I = \int_{-1}^{+1} g(x) dx + E$$

where

$$g(x) = \sum_{i=0}^N \alpha_i(x) f_i + \sum_{i=0}^N \beta_i(x) f_i^{(1)}$$

- Thus

$$I = \int_{-1}^{+1} \left[\sum_{i=0}^N \alpha_i(x) f_i + \sum_{i=0}^N \beta_i(x) f_i^{(1)} \right] dx + E$$

$$\Rightarrow$$

$$I = \sum_{i=0}^N A_i f_i + \sum_{i=0}^N B_i f_i^{(1)} + E$$

where

$$A_i \equiv \int_{-1}^{+1} \alpha_i(x) dx \quad \text{and} \quad B_i \equiv \int_{-1}^{+1} \beta_i(x) dx$$

- Furthermore we can show that

$$E = \int_{-1}^{+1} \left[\frac{p_{N+1}^2(x)}{(2N+2)!} f^{(2N+2)}(x_0) + \text{H.O.T.} \right] dx$$

- Note that we are assuming Taylor series expansions about x_o and using higher order terms in the expansion.
 - Therefore $E = 0$ for any polynomial of degree $2N + 1$ or less!
- The problem that we encounter is that the integration formula as it now stands ***in general*** requires us to know both functional and first derivative values at the nodes!
- Let us select $x_o, x_1, x_2, \dots, x_N$ such that

$$B_i = 0 \quad i = 0, N \quad \Rightarrow$$

$$\int_{-1}^{+1} \beta_i(x) dx = 0 \quad i = 0, N \quad \Rightarrow$$

$$\int_{-1}^{+1} s_i(x) l_{iN}(x) l_{iN}(x) dx = 0 \quad i = 0, N \quad \Rightarrow$$

$$\int_{-1}^{+1} (x - x_i) \frac{p_N(x)}{(x - x_i) p_N^{(1)}(x_i)} l_{iN}(x) dx = 0 \quad i = 0, N \quad \Rightarrow$$

$$\frac{1}{p_N^{(1)}(x_i)} \int_{-1}^{+1} p_N(x) l_{iN}(x) dx = 0 \quad i = 0, N$$

$p_N(x) \Rightarrow$ polynomial of degree $N + 1$

$l_{iN}(x) \Rightarrow$ polynomial of degree N

- Therefore we require $p_N(x)$ to be orthogonal on $[-1, +1]$ to **all** polynomials of degree N or less \Rightarrow any multiple of Legendre-Polynomials will satisfy this.

- Let

$$p_N(x) = \frac{2^{N+1}[(N+1)!]^2}{[2(N+1)]!} P_{N+1}(x)$$

where

$$p_N(x) = (x - x_0)(x - x_1)(x - x_2) \dots (x - x_N)$$

P_{N+1} = the Legendre polynomial of degree $N + 1$

$\frac{2^{N+1}[(N+1)!]^2}{[2(N+1)]!}$ is required to normalize the leading coefficient of $P_{N+1}(x)$

- What have we done by defining $p_N(x)$ in this way \Rightarrow we have selected the integration/interpolation/data points x_0, x_1, \dots, x_N to be the **roots** of $P_{N+1}(x)$.
- In general

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n (x^2 - 1)^n}{dx^n}$$

$$P_0(x) = 1$$

$$P_1(x) = x$$

$$P_2(x) = \frac{1}{2}(3x^2 - 1)$$

$$P_3(x) = \frac{1}{2}(5x^3 - 3x)$$

⋮

- So far we have established
 - Selecting $p_N(x)$ to be proportional to the Legendre Polynomial of degree $N + 1 \Rightarrow$ this satisfies the orthogonality condition which will lead to:

$$\int_{-1}^{+1} \beta_i(x) dx = 0$$

As a result $f_i^{(1)}$ terms will **not** appear in the Gauss-Legendre integration formula.

- If we select $p_N(x)$ to be the Legendre Polynomial of degree $N + 1 \Rightarrow$ the roots of that polynomial will represent the interpolating/integration/data points since $p_N(x) = (x - x_0)(x - x_1) \dots (x - x_N)$ has been set equal to $CP_{N+1}(x)$
- Now we must find the weights of the integration formula. Note that A_i will represent the weights!

$$A_i \equiv \int_{-1}^{+1} \alpha_i(x) dx \quad \Rightarrow$$

$$A_i = \int_{-1}^{+1} t_i(x) l_{iN}(x) l_{iN}(x) dx$$

where

$$t_i(x) = 1 - (x - x_i)^2 l_{iN}^{(1)}(x_i)$$

$$l_{iN}(x) = \frac{p_N(x)}{(x - x_i) p_N^{(1)}(x_i)}$$

$$p_N(x) = (x - x_0) \dots (x - x_N)$$

and where x_0, \dots, x_N are the **roots** of the Legendre polynomial of degree $N + 1$ or

$$p_N(x) = \frac{2^{N+1} [(N+1)!]^2}{[2(N+1)]!} P_{N+1}(x)$$

Two point Gauss-Legendre integration

Develop a 2 point Gauss-Legendre integration formula for $[-1, +1]$. Let

$$g(x) = \sum_{j=0}^1 \alpha_j(x) f_j + \sum_{j=0}^1 \beta_j(x) f_j^{(1)}$$

$$g(x) = \alpha_0(x) f_0 + \alpha_1(x) f_1 + \beta_0(x) f_0^{(1)} + \beta_1(x) f_1^{(1)}$$

- Thus

$$I = \int_{-1}^{+1} g(x) dx + E \quad \Rightarrow$$

$$I = \int_{-1}^{+1} \alpha_0(x) f_0 dx + \int_{-1}^{+1} \alpha_1(x) f_1 dx + \int_{-1}^{+1} \beta_0(x) f_0^{(1)} dx + \int_{-1}^{+1} \beta_1(x) f_1^{(1)} dx \quad \Rightarrow$$

$$I = f_0 \int_{-1}^{+1} \alpha_0(x) dx + f_1 \int_{-1}^{+1} \alpha_1(x) dx + f_0^{(1)} \int_{-1}^{+1} \beta_0(x) dx + f_1^{(1)} \int_{-1}^{+1} \beta_1(x) dx$$

Step 1 - Establish interpolating points

- Interpolation points will be the roots of the Legendre Polynomial of order 2.

$$P_2(x) = \frac{1}{2}(3x^2 - 1) \Rightarrow$$

$$\frac{1}{2}(3x^2 - 1) = 0 \Rightarrow$$

$$3x^2 = 1 \Rightarrow$$

$$x^2 = \frac{1}{3} \Rightarrow$$

$$x_{0,1} = \pm \sqrt{\frac{1}{3}} \Rightarrow$$

$$x_{0,1} = \pm 0.57735$$

- Checking these roots

$$P_2(x) = \frac{1}{2^2 2!} \frac{d^2(x^2 - 1)^2}{dx^2} \Rightarrow$$

$$P_2(x) = \frac{1}{8} \frac{d^2}{dx^2} (x^4 - 2x^2 + 1) \Rightarrow$$

$$P_2(x) = \frac{1}{8} (12x^2 - 4) \Rightarrow$$

$$P_2(x) = \frac{1}{2} (3x^2 - 1)$$

$$p_1(x) = \frac{2^2 (2!)^2}{(2(2))!} P_2(x) \Rightarrow$$

$$p_1(x) = \frac{4 \cdot 4}{4 \cdot 3 \cdot 2} \cdot \frac{1}{2} (3x^2 - 1) \Rightarrow$$

$$p_1(x) = x^2 - \frac{1}{3}$$

- From formula which defines $p_1(x)$ using the integration points

$$p_1(x) = \left(x + \sqrt{\frac{1}{3}}\right)\left(x - \sqrt{\frac{1}{3}}\right) = x^2 - \frac{1}{3}$$

Step 2 - Establish the coefficients of the derivative terms in the integration formula

- Let's demonstrate that with the roots $x_{o,1} = \pm 0.57735$ we will satisfy

$$\int_{-1}^{+1} \beta_o(x) dx = 0 \quad \text{and} \quad \int_{-1}^{+1} \beta_1(x) dx = 0$$

- First develop $\beta_o(x)$ and $\beta_1(x)$ by developing $p_1(x)$, $p_1^{(1)}(x)$, $l_{o1}(x)$, $l_{11}(x)$, $s_o(x)$ and $s_1(x)$

$$p_1(x) = (x - x_o)(x - x_1)$$

$$p_1^{(1)}(x) = (x - x_o) + (x - x_1)$$

$$l_{j1}(x) = \frac{p_1(x)}{(x-x_j)p_1^{(1)}(x_j)} \quad j = 0, 1 \Rightarrow$$

$$l_{j1}(x) = \frac{(x-x_o)(x-x_1)}{(x-x_j)[(x_j-x_o) + (x_j-x_1)]}$$

$$l_{o1}(x) = \frac{(x-x_o)(x-x_1)}{(x-x_o)[x_o-x_o + x_o-x_1]} \Rightarrow$$

$$l_{o1}(x) = \frac{x-x_1}{x_o-x_1}$$

$$l_{11}(x) = \frac{(x-x_o)(x-x_1)}{(x-x_1)[(x_1-x_o) + (x_1-x_1)]} \Rightarrow$$

$$l_{11}(x) = \frac{x-x_o}{x_1-x_o}$$

$$s_o(x) = x - x_o$$

$$s_1(x) = x - x_1$$

- Now we can establish $\beta_o(x)$

$$\beta_o(x) = s_o(x) l_{o1}(x) l_{o1}(x) \quad \Rightarrow$$

$$\beta_o(x) = (x - x_o) \left(\frac{x - x_1}{x_o - x_1} \right) \left(\frac{x - x_1}{x_o - x_1} \right)$$

- Noting that $x_o = -\sqrt{\frac{1}{3}}$, $x_1 = \sqrt{\frac{1}{3}}$

$$\beta_o(x) = \left(x + \sqrt{\frac{1}{3}} \right) \frac{\left(x - \sqrt{\frac{1}{3}} \right) \left(x - \sqrt{\frac{1}{3}} \right)}{\left(-\sqrt{\frac{1}{3}} - \sqrt{\frac{1}{3}} \right)^2} \quad \Rightarrow$$

$$\beta_o(x) = \frac{3}{4} \left[x^3 - \sqrt{\frac{1}{3}} x^2 - \frac{1}{3} x + \left(\frac{1}{3} \right)^{3/2} \right]$$

- Similarly for $\beta_1(x)$

$$\beta_1(x) = s_1(x) l_{11}(x) l_{11}(x) \quad \Rightarrow$$

$$\beta_1(x) = (x - x_1) \frac{(x - x_o)}{(x_1 - x_o)} \cdot \frac{(x - x_o)}{(x_1 - x_o)}$$

- Substituting $x_o = -\sqrt{\frac{1}{3}}$, $x_1 = \sqrt{\frac{1}{3}}$

$$\beta_1(x) = \frac{\left(x - \sqrt{\frac{1}{3}}\right)\left(x + \sqrt{\frac{1}{3}}\right)\left(x + \sqrt{\frac{1}{3}}\right)}{\left(\sqrt{\frac{1}{3}} + \sqrt{\frac{1}{3}}\right)^2} \quad \Rightarrow$$

$$\beta_1(x) = \frac{3}{4} \left[x^3 + \sqrt{\frac{1}{3}} x^2 - \frac{1}{3} x - \left(\frac{1}{3}\right)^{3/2} \right]$$

- Now we can develop $\int_{-1}^{+1} \beta_o(x) dx$

$$\int_{-1}^{+1} \beta_o(x) dx = \int_{-1}^{+1} \frac{3}{4} \left[x^3 - \sqrt{\frac{1}{3}} x^2 - \frac{1}{3} x + \left(\frac{1}{3}\right)^{3/2} \right] dx \Rightarrow$$

$$\int_{-1}^{+1} \beta_o(x) dx = \frac{3}{4} \left[\frac{x^4}{4} - \left(\frac{1}{3}\right)^{3/2} x^3 - \frac{1}{6} x^2 + \left(\frac{1}{3}\right)^{3/2} x \right]_{-1}^{+1} \Rightarrow$$

$$\int_{-1}^{+1} \beta_o(x) dx = \frac{3}{4} \left[\left(\frac{1}{4} - \left(\frac{1}{3}\right)^{3/2} - \frac{1}{6} + \left(\frac{1}{3}\right)^{3/2} \right) - \left(\frac{1}{4} + \left(\frac{1}{3}\right)^{3/2} - \frac{1}{6} - \left(\frac{1}{3}\right)^{3/2} \right) \right] \Rightarrow$$

$$\int_{-1}^{+1} \beta_o(x) dx = \mathbf{0}$$

- Develop $\int_{-1}^{+1} \beta_1(x) dx$

$$\int_{-1}^{+1} \beta_1(x) dx = \int_{-1}^{+1} \frac{3}{4} \left[x^3 + \sqrt{\frac{1}{3}} x^2 - \frac{1}{3} x - \left(\frac{1}{3}\right)^{3/2} \right] dx \Rightarrow$$

$$\int_{-1}^{+1} \beta_1(x) dx = \frac{3}{4} \left[\frac{x^4}{4} + \left(\frac{1}{3}\right)^{3/2} x^3 - \frac{1}{6} x^2 - \left(\frac{1}{3}\right)^{3/2} x \right]_{-1}^{+1} \Rightarrow$$

$$\int_{-1}^{+1} \beta_1(x) dx = \frac{3}{4} \left[\left(\frac{1}{4} + \left(\frac{1}{3}\right)^{3/2} - \frac{1}{6} - \left(\frac{1}{3}\right)^{3/2} \right) - \left(\frac{1}{4} - \left(\frac{1}{3}\right)^{3/2} - \frac{1}{6} + \left(\frac{1}{3}\right)^{3/2} \right) \right] \Rightarrow$$

$$\int_{-1}^{+1} \beta_1(x) dx = \mathbf{0}$$

- Now our integration formula reduces to:

$$I = f_o \int_{-1}^{+1} \alpha_o(x) dx + f_1 \int_{-1}^{+1} \alpha_1(x) dx \quad \Rightarrow$$

$$I = A_o f_o + A_1 f_1$$

where

$$A_o \equiv \int_{-1}^{+1} \alpha_o(x) dx \quad \text{and} \quad A_1 \equiv \int_{-1}^{+1} \alpha_1(x) dx$$

Step 3 - Develop A_o, A_1

- Establish $\alpha_o(x)$

$$\alpha_o(x) = t_o(x) l_{o1}(x) l_{o1}(x) \quad \Rightarrow$$

$$\alpha_o(x) = [1 - (x - x_o) 2l_{o1}^{(1)}(x_o)] l_{o1}(x) l_{o1}(x) \quad \Rightarrow$$

$$\alpha_o(x) = \left\{ 1 - (x - x_o) \left[\frac{2}{x_o - x_1} \right] \right\} \left(\frac{x - x_1}{x_o - x_1} \right) \left(\frac{x - x_1}{x_o - x_1} \right) \Rightarrow$$

$$\alpha_o(x) = \left\{ 1 - \left(x + \sqrt{\frac{1}{3}} \right) \left[\frac{2}{-\sqrt{\frac{1}{3}} - \sqrt{\frac{1}{3}}} \right] \right\} \frac{\left(x - \sqrt{\frac{1}{3}} \right) \left(x - \sqrt{\frac{1}{3}} \right)}{\left(-\sqrt{\frac{1}{3}} - \sqrt{\frac{1}{3}} \right) \left(-\sqrt{\frac{1}{3}} - \sqrt{\frac{1}{3}} \right)} \Rightarrow$$

$$\alpha_o(x) = \frac{3}{4} \left\{ 1 - \left(x + \sqrt{\frac{1}{3}} \right) \left(\frac{2}{-2\sqrt{\frac{1}{3}}} \right) \right\} \left(x^2 - 2\sqrt{\frac{1}{3}}x + \frac{1}{3} \right) \Rightarrow$$

$$\alpha_o(x) = \frac{3\sqrt{3}}{4} \left\{ \frac{2}{\sqrt{3}} + x \right\} \left(x^2 - 2\sqrt{\frac{1}{3}}x + \frac{1}{3} \right) \Rightarrow$$

$$\alpha_o(x) = \frac{3}{4}\sqrt{3} \left\{ x^3 - x + 2\left(\frac{1}{3}\right)^{3/2} \right\}$$

- Develop $\int_{-1}^{+1} \alpha_o(x) dx$

$$\int_{-1}^{+1} \alpha_o(x) dx = \int_{-1}^{+1} \left[\frac{3}{4} \sqrt{3} \left(x^3 - x + 2 \left(\frac{1}{3} \right)^{3/2} \right) \right] dx \Rightarrow$$

$$\int_{-1}^{+1} \alpha_o(x) dx = \frac{3}{4} \sqrt{3} \left[\frac{x^4}{4} - \frac{x^2}{2} + 2 \left(\frac{1}{3} \right)^{3/2} x \right]_{-1}^{+1} \Rightarrow$$

$$\int_{-1}^{+1} \alpha_o(x) dx = \frac{3}{4} \sqrt{3} \left[\left(\frac{1}{4} - \frac{1}{2} + 2 \left(\frac{1}{3} \right)^{3/2} \right) - \left(\frac{1}{4} - \frac{1}{2} - 2 \left(\frac{1}{3} \right)^{3/2} \right) \right] \Rightarrow$$

$$\int_{-1}^{+1} \alpha_o(x) dx = \frac{3}{4} \sqrt{3} \left(4 \sqrt{\frac{1}{3}} \frac{1}{3} \right) \Rightarrow$$

$$\int_{-1}^{+1} \alpha_o(x) dx = 1 \Rightarrow$$

$$\mathbf{A_o = 1}$$

- Similarly we can show that $A_1 = \int_{-1}^{+1} \alpha_1(x) dx = 1$

- **Thus we have established the two point Gauss Quadrature rule**

$$I = \int_{-1}^{+1} f(x) dx = w_0 f_0 + w_1 f_1$$

where $x_0 = -\sqrt{\frac{1}{3}}$ and $x_1 = +\sqrt{\frac{1}{3}}$ are the integration points and $w_0 = w_1 = 1$

- We note that this integration rule was established by defining a Hermite cubic interpolating function and defining the integration points x_0, x_1 such that

$$\int_{-1}^{+1} \beta_0(x) dx = 0 \quad \text{and} \quad \int_{-1}^{+1} \beta_1(x) dx = 0$$

- Therefore the functional derivative values drop out of the Gauss Legendre integration formula!