# A FIRST ORDER DIVIDED DIFFERENCE

For a given function $f(x)$ and two distinct points $x_0$ and $x_1$, define

$$f[x_0, x_1] = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

This is called a first order divided difference of $f(x)$.

By the Mean-value theorem,

$$f(x_1) - f(x_0) = f'(c)\,(x_1 - x_0)$$

for some $c$ between $x_0$ and $x_1$. Thus

$$f[x_0, x_1] = f'(c)$$

and the divided difference in very much like the derivative, especially if $x_0$ and $x_1$ are quite close together. In fact,

$$f'\left(\frac{x_1 + x_0}{2}\right) \approx f[x_0, x_1]$$

is quite an accurate approximation of the derivative (see §7.4).

# SECOND ORDER DIVIDED DIFFERENCES

Given three distinct points $x_0$, $x_1$, and $x_2$, define

$$f[x_0, x_1, x_2] = \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0}$$

This is called the <u>second order divided difference</u> of $f(x)$.

By a fairly complicated argument, we can show

$$f[x_0, x_1, x_2] = \frac{1}{2} f''(c)$$

for some $c$ intermediate to $x_0$, $x_1$, and $x_2$. In fact, as we investigate in §7.4,

$$f''(x_1) \approx 2f[x_0, x_1, x_2]$$

in the case the nodes are evenly spaced,

$$x_1 - x_0 = x_2 - x_1$$

# EXAMPLE

Consider the table

| $x$ | 1 | 1.1 | 1.2 | 1.3 | 1.4 |
|---|---|---|---|---|---|
| $\cos x$ | .54030 | .45360 | .36236 | .26750 | .16997 |

Let $x_0 = 1$, $x_1 = 1.1$, and $x_2 = 1.2$. Then

$$f[x_0, x_1] = \frac{.45360 - .54030}{1.1 - 1} = -.86700$$
$$f[x_1, x_2] = \frac{.36236 - .45360}{1.1 - 1} = -.91240$$

$$f[x_0, x_1, x_2] = \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0}$$
$$= \frac{-.91240 - (-.86700)}{1.2 - 1.0} = -.22700$$

For comparison,

$$f'\left(\frac{x_1 + x_0}{2}\right) = -\sin(1.05) = -.86742$$
$$\frac{1}{2} f''(x_1) = -\cos(1.1) = -.22680$$

# GENERAL DIVIDED DIFFERENCES

Given $n+1$ distinct points $x_0, ..., x_n$, with $n \geq 2$, define

$$f[x_0, ..., x_n] = \frac{f[x_1, ..., x_n] - f[x_0, ..., x_{n-1}]}{x_n - x_0}$$

This is a recursive definition of the $n^{\text{th}}$-order divided difference of $f(x)$, using divided differences of order $n$. Its relation to the derivative is as follows:

$$f[x_0, ..., x_n] = \frac{1}{n!} f^{(n)}(c)$$

for some $c$ intermediate to the points $\{x_0, ..., x_n\}$. Let $I$ denote the interval

$$I = [\min\{x_0, ..., x_n\}, \max\{x_0, ..., x_n\}]$$

Then $c \in I$, and the above result is based on the assumption that $f(x)$ is $n$-times continuously differentiable on the interval $I$.

# EXAMPLE

The following table gives divided differences for the data in

| $x$ | 1 | 1.1 | 1.2 | 1.3 | 1.4 |
|---|---|---|---|---|---|
| $\cos x$ | .54030 | .45360 | .36236 | .26750 | .16997 |

For the column headings, we use

$$D^k f(x_i) = f[x_i, ..., x_{i+k}]$$

| $i$ | $x_i$ | $f(x_i)$ | $Df(x_i)$ | $D^2 f(x_i)$ | $D^3 f(x_i)$ | $D^4 f(x_i)$ |
|---|---|---|---|---|---|---|
| 0 | 1.0 | .54030 | -.8670 | -.2270 | .1533 | .0125 |
| 1 | 1.1 | .45360 | -.9124 | -.1810 | .1583 | |
| 2 | 1.2 | .36236 | -.9486 | -.1335 | | |
| 3 | 1.3 | .26750 | -.9753 | | | |
| 4 | 1.4 | .16997 | | | | |

These were computed using the recursive definition

$$f[x_0, ..., x_n] = \frac{f[x_1, ..., x_n] - f[x_0, ..., x_{n-1}]}{x_n - x_0}$$

## ORDER OF THE NODES

Looking at $f[x_0, x_1]$, we have

$$f[x_0, x_1] = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \frac{f(x_0) - f(x_1)}{x_0 - x_1} = f[x_1, x_0]$$

The order of $x_0$ and $x_1$ does not matter. Looking at

$$f[x_0, x_1, x_2] = \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0}$$

we can expand it to get

$$f[x_0, x_1, x_2] = \frac{f(x_0)}{(x_0 - x_1)(x_0 - x_2)} + \frac{f(x_1)}{(x_1 - x_0)(x_1 - x_2)} + \frac{f(x_2)}{(x_2 - x_0)(x_2 - x_1)}$$

With this formula, we can show that the order of the arguments $x_0, x_1, x_2$ does not matter in the final value of $f[x_0, x_1, x_2]$ we obtain. Mathematically,

$$f[x_0, x_1, x_2] = f[x_{i_0}, x_{i_1}, x_{i_2}]$$

for any permutation $(i_0, i_1, i_2)$ of $(0, 1, 2)$.

We can show in general that the value of $f[x_0, ..., x_n]$ is independent of the order of the arguments $\{x_0, ..., x_n\}$, even though the intermediate steps in its calculations using

$$f[x_0, ..., x_n] = \frac{f[x_1, ..., x_n] - f[x_0, ..., x_{n-1}]}{x_n - x_0}$$

are order dependent.

We can show

$$f[x_0, ..., x_n] = f[x_{i_0}, ..., x_{i_n}]$$

for any permutation $(i_0, i_1, ..., i_n)$ of $(0, 1, ..., n)$.

## COINCIDENT NODES

What happens when some of the nodes $\{x_0, ..., x_n\}$ are not distinct. Begin by investigating what happens when they all come together as a single point $x_0$.

For first order divided differences, we have

$$\lim_{x_1 \to x_0} f[x_0, x_1] = \lim_{x_1 \to x_0} \frac{f(x_1) - f(x_0)}{x_1 - x_0} = f'(x_0)$$

We extend the definition of $f[x_0, x_1]$ to coincident nodes using

$$f[x_0, x_0] = f'(x_0)$$

For second order divided differences, recall

$$f[x_0, x_1, x_2] = \frac{1}{2}f''(c)$$

with $c$ intermediate to $x_0$, $x_1$, and $x_2$.

Then as $x_1 \longrightarrow x_0$ and $x_2 \longrightarrow x_0$, we must also have that $c \longrightarrow x_0$. Therefore,

$$\lim_{\substack{x_1 \to x_0 \\ x_2 \to x_0}} f[x_0, x_1, x_2] = \frac{1}{2}f''(x_0)$$

We therefore define

$$f[x_0, x_0, x_0] = \frac{1}{2}f''(x_0)$$

For the case of general $f[x_0, ..., x_n]$, recall that

$$f[x_0, ..., x_n] = \frac{1}{n!} f^{(n)}(c)$$

for some $c$ intermediate to $\{x_0, ..., x_n\}$. Then

$$\lim_{\{x1,...,x_n\} \to x_0} f[x_0, ..., x_n] = \frac{1}{n!} f^{(n)}(x_0)$$

and we define

$$f[\underbrace{x_0, ..., x_0}_{n+1 \text{ times}}] = \frac{1}{n!} f^{(n)}(x_0)$$

What do we do when only some of the nodes are coincident. This too can be dealt with, although we do so here only by examples.

$$\begin{aligned} f[x_0, x_1, x_1] &= \frac{f[x_1, x_1] - f[x_0, x_1]}{x_1 - x_0} \\ &= \frac{f'(x_1) - f[x_0, x_1]}{x_1 - x_0} \end{aligned}$$

The recursion formula can be used in general in this way to allow all possible combinations of possibly coincident nodes.

# LAGRANGE'S FORMULA FOR THE INTERPOLATION POLYNOMIAL

Recall the general interpolation problem: find a polynomial $P_n(x)$ for which

$$\begin{gathered} \text{deg}(P_n) \le n \\ P_n(x_i) = y_i, \qquad i = 0, 1, \cdots, n \end{gathered}$$

with given data points

$$(x_0, y_0), (x_1, y_1), \cdots, (x_n, y_n)$$

and with $\{x_0, ..., x_n\}$ distinct points.

In §5.1, we gave the solution as <u>Lagrange's formula</u>

$$P_n(x) = y_0 L_0(x) + y_1 L_1(x) + \cdots + y_n L_n(x)$$

with $\{L_0(x), ..., L_n(x)\}$ the Lagrange basis polynomials. Each $L_j$ is of degree $n$ and it satisfies

$$L_j(x_i) = \begin{cases} 1, & j = i \\ 0, & j \ne i \end{cases}$$

for $i = 0, 1, ..., n$.

# THE NEWTON DIVIDED DIFFERENCE FORM OF THE INTERPOLATION POLYNOMIAL

Let the data values for the problem

$$\begin{gathered}\deg(P_n) \le n \\ P_n(x_i) = y_i, \qquad i = 0, 1, \cdots, n\end{gathered}$$

be generated from a function $f(x)$:

$$y_i = f(x_i), \qquad i = 0, 1, ..., n$$

Using the divided differences

$$f[x_0, x_1],\ f[x_0, x_1, x_2], ..., f[x_0, ..., x_n]$$

we can write the interpolation polynomials

$$P_1(x),\ P_2(x), ..., P_n(x)$$

in a way that is simple to compute.

$$\begin{aligned}P_1(x) &= f(x_0) + f[x_0, x_1]\,(x - x_0) \\ P_2(x) &= f(x_0) + f[x_0, x_1]\,(x - x_0) \\ &\quad + f[x_0, x_1, x_2]\,(x - x_0)\,(x - x_1) \\ &= P_1(x) + f[x_0, x_1, x_2]\,(x - x_0)\,(x - x_1)\end{aligned}$$

For the case of the general problem

$$\deg(P_n) \leq n$$
$$P_n(x_i) = y_i, \qquad i = 0, 1, \cdots, n$$

we have

$$\begin{aligned}
P_n(x) = \; & f(x_0) + f[x_0, x_1]\,(x - x_0) \\
& + f[x_0, x_1, x_2]\,(x - x_0)\,(x - x_1) \\
& + f[x_0, x_1, x_2, x_3]\,(x - x_0)\,(x - x_1)\,(x - x_2) \\
& + \cdots \\
& + f[x_0, ..., x_n]\,(x - x_0) \cdots (x - x_{n-1})
\end{aligned}$$

From this we have the recursion relation

$$P_n(x) = P_{n-1}(x) + f[x_0, ..., x_n]\,(x - x_0) \cdots (x - x_{n-1})$$

in which $P_{n-1}(x)$ interpolates $f(x)$ at the points in $\{x_0, ..., x_{n-1}\}$.

Example: Recall the table

| $i$ | $x_i$ | $f(x_i)$ | $Df(x_i)$ | $D^2f(x_i)$ | $D^3f(x_i)$ | $D^4f(x_i)$ |
|---|---|---|---|---|---|---|
| 0 | 1.0 | .54030 | -.8670 | -.2270 | .1533 | .0125 |
| 1 | 1.1 | .45360 | -.9124 | -.1810 | .1583 | |
| 2 | 1.2 | .36236 | -.9486 | -.1335 | | |
| 3 | 1.3 | .26750 | -.9753 | | | |
| 4 | 1.4 | .16997 | | | | |

with $D^k f(x_i) = f[x_i, ..., x_{i+k}], \quad k = 1, 2, 3, 4$. Then

$$
\begin{aligned}
P_1(x) &= .5403 - .8670\,(x-1) \\
P_2(x) &= P_1(x) - .2270\,(x-1)\,(x-1.1) \\
P_3(x) &= P_2(x) + .1533\,(x-1)\,(x-1.1)\,(x-1.2) \\
P_4(x) &= P_3(x) \\
&\quad +.0125\,(x-1)\,(x-1.1)\,(x-1.2)\,(x-1.3)
\end{aligned}
$$

Using this table and these formulas, we have the following table of interpolants for the value $x = 1.05$. The true value is $\cos(1.05) = .49757105$.

| $n$ | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| $P_n(1.05)$ | .49695 | .49752 | .49758 | .49757 |
| $Error$ | 6.20E−4 | 5.00E−5 | −1.00E−5 | 0.0 |

# EVALUATION OF THE DIVIDED DIFFERENCE INTERPOLATION POLYNOMIAL

Let

$$\begin{aligned} d_1 &= f[x_0, x_1] \\ d_2 &= f[x_0, x_1, x_2] \\ &\vdots \\ d_n &= f[x_0, ..., x_n] \end{aligned}$$

Then the formula

$$\begin{aligned} P_n(x) = {} & f(x_0) + f[x_0, x_1]\,(x - x_0) \\ & + f[x_0, x_1, x_2]\,(x - x_0)\,(x - x_1) \\ & + f[x_0, x_1, x_2, x_3]\,(x - x_0)\,(x - x_1)\,(x - x_2) \\ & + \cdots \\ & + f[x_0, ..., x_n]\,(x - x_0) \cdots (x - x_{n-1}) \end{aligned}$$

can be written as

$$\begin{aligned} P_n(x) = {} & f(x_0) + (x - x_0)\,(d_1 + (x - x_1)\,(d_2 + \cdots \\ & + (x - x_{n-2})\,(d_{n-1} + (x - x_{n-1})\,d_n) \cdots ) \end{aligned}$$

Thus we have a nested polynomial evaluation, and this is quite efficient in computational cost.