# NEWTON'S METHOD

For a general equation $f(x) = 0$, we assume we are given an initial estimate $x_0$ of the root $\alpha$. The iterates are generated by the formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \qquad n = 0, 1, 2, ...$$

As an example, consider solving

$$f(x) \equiv x^6 - x - 1 = 0$$

for its positive root $\alpha$. An initial guess $x_0$ can be generated from a graph of $y = f(x)$.

As seen from the output, the convergence is very rapid. The iterate $x_6$ is accurate to the machine precision of around 16 decimal digits. This is the typical behaviour seen with Newton's method for most problems, but not all.

We could also have considered the problem of solving the annuity equation

$$f(x) \equiv 1000\left[\left(1+\frac{x}{12}\right)^{480}-1\right] \\ -5000\left[1-\left(1+\frac{x}{12}\right)^{-240}\right]=0$$

However, it turns out that you have to be very close to the root in this case in order to get good convergence. This phenomena is discussed further at a later time; and the bisection method is preferable in this instance.

# AN ERROR FORMULA

Suppose we use Taylor's formula to expand $f(\alpha)$ about $x = x_n$. Then we have

$$f(\alpha) = f(x_n) + (\alpha - x_n)\, f'(x_n) + \frac{1}{2}\,(\alpha - x_n)^2\, f''(c_n)$$

for some $c_n$ between $\alpha$ and $x_n$. Note that $f(\alpha) = 0$. Then divide both sides of this equation by $f'(x_n)$, yielding

$$0 = \frac{f(x_n)}{f'(x_n)} + \alpha - x_n + (\alpha - x_n)^2\, \frac{f''(c_n)}{2f'(x_n)}$$

Note that

$$\frac{f(x_n)}{f'(x_n)} - x_n = -x_{n+1}$$

and thus

$$\alpha - x_{n+1} = -\frac{f''(c_n)}{2f'(x_n)}\,(\alpha - x_n)^2$$

For $x_n$ close to $\alpha$, and therefore $c_n$ also close to $\alpha$, we have

$$\alpha - x_{n+1} \approx -\frac{f''(\alpha)}{2f'(\alpha)} (\alpha - x_n)^2$$

Thus Newton's method is quadratically convergent, provided $f'(\alpha) \neq 0$ and $f(x)$ is twice differentiable in the vicinity of the root $\alpha$.

We can also use this to explore the 'interval of convergence' of Newton's method. Write the above as

$$\alpha - x_{n+1} \approx M (\alpha - x_n)^2 , \qquad M = -\frac{f''(\alpha)}{2f'(\alpha)}$$

Multiply both sides by $M$ to get

$$M (\alpha - x_{n+1}) \approx [M (\alpha - x_n)]^2$$

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Then we want these quantities to decrease; and this suggests choosing $x_0$ so that

$$|M(\alpha - x_0)| < 1$$

$$|\alpha - x_0| < \frac{1}{|M|} = \left| \frac{2f'(\alpha)}{f''(\alpha)} \right|$$

If $|M|$ is very large, then we may need to have a very good initial guess in order to have the iterates $x_n$ converge to $\alpha$.

## ADVANTAGES & DISADVANTAGES

Advantages: 1. It is rapidly convergent in most cases.
2. It is simple in its formulation, and therefore relatively easy to apply and program.
3. It is intuitive in its construction. This means it is easier to understand its behaviour, when it is likely to behave well and when it may behave poorly.

Disadvantages: 1. It may not converge.
2. It has likely to have difficulty if $f'(\alpha) = 0$. This means the $x$-axis is tangent to the graph of $y = f(x)$ at $x = \alpha$.
3. It needs to know both $f(x)$ and $f'(x)$. Contrast this with the bisection method which requires only $f(x)$.