Romberg Integration

By making use of two successive applications of the trapezoidal rule in which the step size $h = (b-a)/n$ is progressively decreased, we can get a numerical integration algorithm with $O(h^4)$ truncation error rather than $O(h^2)$. To see how this is done, note that the true value of the integral can be written as

$$I = I(h_1) + E(h_1) = I(h_2) + E(h_2), \qquad (1)$$

where $I(h_1)$ and $E(h_1)$ represent, respectively, the estimate and error after the first application of the trapezoidal rule with step size h_1, and $I(h_2)$ and $E(h_2)$ represent the estimate and error after the second application with step size h_2. Now, recall that

$$E(h_1) \approx \frac{b-a}{12} h_1^2 \bar{f}'',$$

and

$$E(h_2) \approx \frac{b-a}{12} h_2^2 \bar{f}'',$$

where we make the assumption that $\bar{f}''$ is approximately constant regardless of the step size. Hence, we have

$$\frac{E(h_1)}{E(h_2)} \approx \left(\frac{h_1}{h_2}\right)^2,$$

or

$$E(h_1) \approx E(h_2)\left(\frac{h_1}{h_2}\right)^2.$$

Substituting this expression back into Equation (1), we get

$$I(h_1) + E(h_2)\left(\frac{h_1}{h_2}\right)^2 = I(h_2) + E(h_2),$$

which can be solved for

$$E(h_2) \approx \frac{I(h_1) - I(h_2)}{1 - (h_1/h_2)^2}.$$

Finally, substituting into Equation (1) again, we get

$$I = I(h_2) + E(h_2) \approx I(h_2) + \frac{I(h_2) - I(h_1)}{(h_1/h_2)^2 - 1}.$$

This procedure is called *Richardson's extrapolation*. In the special case that the step size is reduced by a factor of 2, Richardson's extrapolation becomes

$$I \approx \frac{4}{3} I(h_2) - \frac{1}{3} I(h_1). \qquad (2)$$

It turns out that this approximation has a truncation error of order $O(h^4)$ rather than just $O(h^2)$.

Example: Consider the integral

$$I = \int_0^{0.8} \left(0.2 + 25x - 200x^2 + 675x^3 - 900x^4 + 400x^5\right) dx.$$

Three successive applications of the trapezoidal rule gives the following table of results

n	h	$I(h)$	ε_t
1	0.8	0.1728	89.5
2	0.4	1.0688	34.9
4	0.2	1.4848	9.5

We can combine the first two estimates with step sizes $h = 0.8$ and $h = 0.4$ using Equation (2) to give a new estimate as follows:

$$\begin{aligned} I &\approx \frac{4}{3} I(0.4) - \frac{1}{3} I(0.8) \\ &= \frac{4}{3}(1.0688) - \frac{1}{3}(0.1728) \\ &\approx 1.367467. \end{aligned}$$

This new estimate has a relative truncation error of $\varepsilon_t = 16.6\%$, which is considerably better than either of the estimates used to generate it. Similarly, we can combine the second two estimates to give another new estimate in the following manner:

$$\begin{aligned} I &\approx \frac{4}{3} I(0.2) - \frac{1}{3} I(0.4) \\ &= \frac{4}{3}(1.4848) - \frac{1}{3}(1.0688) \\ &\approx 1.623467. \end{aligned}$$

This estimate has a relative truncation error of only $\varepsilon_t = 1.0\%$, which is again better than either of the estimates used to generate it.

Actually, we can continue this procedure by using a sequence of improved (or extrapolated) estimates rather than the sequence of original estimates based on the trapezoidal rule. For example, we can combine two $O\left(h^4\right)$ estimates to produce an $O\left(h^6\right)$ estimate using the formula

$$I \approx \frac{16}{15} I_m - \frac{1}{15} I_l,$$

where I_m and I_l are the *more* and *less* accurate $O\left(h^4\right)$ estimates, respectively. Using the results of the previous example, this procedure gives

$$I \approx \frac{16}{15}(1.623467) - \frac{1}{15} I_l(1.367467) = 1.640533,$$

which is correct to seven significant digits.

The iterative procedure described above is known as *Romberg integration*, and the general formula is given by

$$I_{j,k} = \frac{4^{k-1} I_{j+1,k-1} - I_{j,k-1}}{4^{k-1} - 1},$$

where $I_{j+1,k-1}$ and $I_{j,k-1}$ represent the more and less accurate integrals, respectively, and $I_{j,k}$ represents the improved estimate. The index k represents the level of integration, where $k = 1$ corresponds to the original trapezoidal-rule $O\left(h^2\right)$ estimates, $k = 2$ corresponds to the $O\left(h^4\right)$ estimates, $k = 3$ corresponds to the $O\left(h^6\right)$ estimates, *etc*. This Romberg integration procedure is illustrated below using the results from our previous example plus one additional step in the algorithm.

j	$I_{j,1}$	$I_{j,2}$	$I_{j,3}$	$I_{j,4}$
1	0.172800	1.367467	1.640533	1.640533
2	1.068800	1.623467	1.640533	
3	1.484800	1.639467		
4	1.600800			

Note that we can form estimates of the relative truncation error at the end of any sequence of estimates from the Romberg integration procedure using the following formula

$$\varepsilon_a = \left| \frac{I_{1,k} - I_{1,k-1}}{I_{1,k}} \right|$$

In general, Romberg integration gives better results than using a Simpson's rule with far fewer function evaluations. This not only saves computation time, but also reduces the accumulated round-off error (propagated error) in the final result.