THE SECANT METHOD

Newton's method was based on using the line tangent to the curve of $y = f(x)$, with the point of tangency $(x_0, f(x_0))$. When $x_0 \approx \alpha$, the graph of the tangent line is approximately the same as the graph of $y = f(x)$ around $x = \alpha$. We then used the root of the tangent line to approximate α.

Consider using an approximating line based on 'interpolation'. We assume we have <u>two</u> estimates of the root α, say x_0 and x_1. Then we produce a linear function

$$q(x) = a_0 + a_1 x$$

with

$$q(x_0) = f(x_0), \qquad q(x_1) = f(x_1) \qquad (**)$$

This line is sometimes called a <u>secant line</u>. Its equation is given by

$$q(x) = \frac{(x_1 - x)\, f(x_0) + (x - x_0)\, f(x_1)}{x_1 - x_0}$$

This is linear in x; and by direction evaluation, it satisfies the interpolation conditions of (**). Two possibilities are shown in the graphs in Figures 4.3 and 4.4 in the text (page 79).

We now solve the equation $q(x) = 0$, denoting the root by x_2. This yields

$$x_2 = x_1 - f(x_1) \div \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

We can now repeat the process. Use x_1 and x_2 to produce another secant line, and then uses its root to approximate α. This yields the general iteration formula

$$x_{n+1} = x_n - f(x_n) \div \frac{f(x_n) - f(x_{n-1})}{x_n - x_{n-1}}, \qquad n = 1, 2, 3...$$

This is called the <u>secant method</u> for solving $f(x) = 0$.

A numerical example for solving

$$x^6 - x - 1 = 0$$

is given in the text, on page 81. This is the same equation as used earlier to illustrate the Newton method. It is clear from the numerical results that the secant method requires more iterates than the Newton method; but note that the secant method does not require a knowledge of $f'(x)$.

Also, note that the secant method can be considered an approximation of the Newton method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

by using the approximation

$$f'(x_n) \approx \frac{f(x_n) - f(x_{n-1})}{x_n - x_{n-1}}$$

CONVERGENCE ANALYSIS

With a combination of algebraic manipulation and the mean-value theorem from calculus, we can show

$$\alpha - x_{n+1} = (\alpha - x_n)(\alpha - x_{n-1})\left[\frac{-f''(\xi_n)}{2f'(\zeta_n)}\right], \qquad (*)$$

with ξ_n and ζ_n unknown points. The point ξ_n is located between the minimum and maximum of x_{n-1}, x_n, and α; and ζ_n is located between the minimum and maximum of x_{n-1} and x_n. Recall for Newton's method that the Newton iterates satisfied

$$\alpha - x_{n+1} = (\alpha - x_n)^2\left[\frac{-f''(\xi_n)}{2f'(x_n)}\right]$$

which closely resembles (*) above.

Using (*), it can be shown that x_n converges to α, and moreover,

$$\lim_{n\to\infty} \frac{|\alpha - x_{n+1}|}{|\alpha - x_n|^r} = \left|\frac{f''(\alpha)}{2f'(\alpha)}\right|^{r-1} \equiv c$$

where $\frac{1}{2}(1 + \text{sqrt}(5)) \doteq 1.62$. This assumes that x_0 and x_1 are chosen sufficiently close to α; and how close this is will vary with the function f. In addition, the above result assumes $f(x)$ has two continuous derivatives for all x in some interval about α.

The above says that when we are close to α, that

$$|\alpha - x_{n+1}| \approx c\,|\alpha - x_n|^r$$

This looks very much like the Newton result

$$\alpha - x_{n+1} \approx M\,(\alpha - x_n)^2, \qquad M = \frac{-f''(\alpha)}{2f'(\alpha)}$$

and $c = |M|^{r-1}$. Both the secant and Newton methods converge at faster than a linear rate, and they are called <u>superlinear</u> methods.

The secant method converge slower than Newton's method; but it is still quite rapid. It is rapid enough that we can prove

$$\lim_{n\to\infty} \frac{|x_{n+1} - x_n|}{|\alpha - x_n|} = 1$$

and therefore,

$$|\alpha - x_n| \approx |x_{n+1} - x_n|$$

is a good error estimator.

A note of warning: Do not combine the secant formula and write it in the form

$$x_{n+1} = \frac{f(x_n)x_{n-1} - f(x_{n-1})x_n}{f(x_n) - f(x_{n-1})}$$

This has enormous loss of significance errors as compared with the earlier formulation.

COSTS OF SECANT & NEWTON METHODS

The Newton method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \qquad n = 0, 1, 2, ...$$

requires two function evaluations per iteration, that of $f(x_n)$ and $f'(x_n)$. The secant method

$$x_{n+1} = x_n - f(x_n) \div \frac{f(x_n) - f(x_{n-1})}{x_n - x_{n-1}}, \qquad n = 1, 2, 3...$$

requires 1 function evaluation per iteration, following the initial step.

For this reason, the secant method is often faster in time, even though more iterates are needed with it than with Newton's method to attain a similar accuracy.

ADVANTAGES & DISADVANTAGES

Advantages of secant method: 1. It converges at faster than a linear rate, so that it is more rapidly convergent than the bisection method.
2. It does not require use of the derivative of the function, something that is not available in a number of applications.
3. It requires only one function evaluation per iteration, as compared with Newton's method which requires two.

Disadvantages of secant method: 1. It may not converge.
2. There is no guaranteed error bound for the computed iterates.
3. It is likely to have difficulty if $f'(\alpha) = 0$. This means the x-axis is tangent to the graph of $y = f(x)$ at $x = \alpha$.
4. Newton's method generalizes more easily to new methods for solving simultaneous systems of nonlinear equations.

BRENT'S METHOD

Richard Brent devised a method combining the advantages of the bisection method and the secant method.

1. It is guaranteed to converge.
2. It has an error bound which will converge to zero in practice.
3. For most problems $f(x) = 0$, with $f(x)$ differentiable about the root α, the method behaves like the secant method.
4. In the worst case, it is not too much worse in its convergence than the bisection method.

In Matlab, it is implemented as $fzero$; and it is present in most Fortran numerical analysis libraries. I have put a double precision Fortran implementation of the method, called simply zero.f, into the class account.