Simpson's Rule (1/3)

Simpson's rule is a numerical integration technique which is based on the use of parabolic arcs to approximate $f(x)$ instead of the straight lines employed as the interpolating polynomials in the trapezoidal rule. Higher order polynomials, such as cubics, can also be used to obtain more accurate results. Consider the integral of a function $f(x)$ over an interval $a \le b$

$$I = \int_a^b f(x)dx \tag{14}$$

Simpson's 1/3 rule is obtained when a second-order interpolating polynomial is substituted for $f(x)$

$$I = \int_a^b f(x)dx \approx \int_a^b f_2(x)dx \tag{15}$$

where $f_2(x)$ is a second-order Lagrange interpolating polynomial. Consider an expanded view of a general region including one panel as shown in the following Figure.

In the Figure, the points $f(x_{i-1})$, $f(x_i)$, and $f(x_{i+1})$ have been connected by a parabola. This parabola approximates the function $f(x)$ between x_{i-1} and x_{i+1}. Approximating the area of the panel by the area under the parabola yields

$$\begin{aligned}\int_{x_{i-1}}^{x_{i+1}} f_2(x)dx = \int_{x_{i-1}}^{x_{i+1}} \Big[& \frac{(x-x_i)(x-x_{i+1})}{(x_{i-1}-x_i)(x_{i-1}-x_{i+1})} f(x_{i-1}) + \frac{(x-x_{i-1})(x-x_{i+1})}{(x_i-x_{i-1})(x_i-x_{i+1})} f(x_i) \\ & + \frac{(x-x_{i-1})(x-x_i)}{(x_{i+1}-x_{i-1})(x_{i+1}-x_i)} f(x_{i+1}) \Big]dx\end{aligned} \tag{16}$$

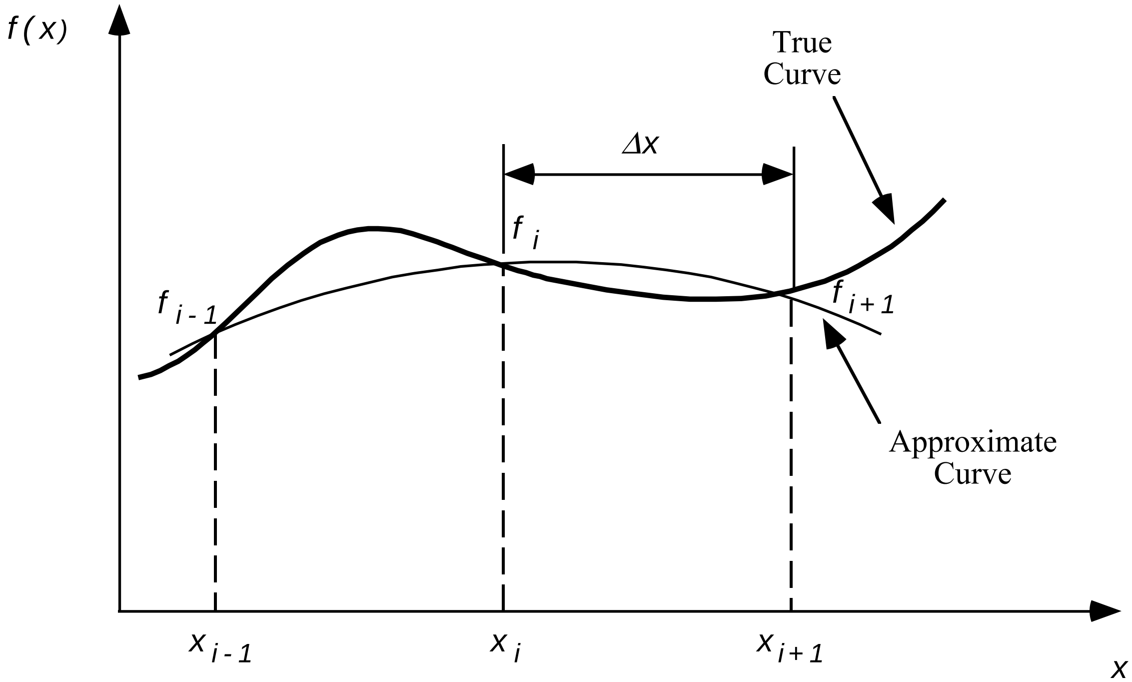

Figure. Two panels for Simpson's rule.

This expression can be integrated and simplified to

$$\int_{x_{i-1}}^{x_{i+1}} f_2(x)dx \;=\; \frac{\Delta x}{3}\left[f(x_{i-1})+4\,f(x_i)+f(x_{i+1})\right] \tag{17}$$

By extending equation (6), the Simpson's rule approximation to the integral over the entire interval is

$$\int_a^b f(x)dx \;\approx\; \frac{\Delta x}{3}(f_0+4\,f_1+2\,f_2+4\,f_3+\cdots+2\,f_{n-2}+4\,f_{n-1}+\,f_n) \tag{18}$$

or

$$\int_a^b f(x)dx \;\approx\; \frac{\Delta x}{3}\Big(f_0+4\sum_{\substack{i=1\\ i\ odd}}^{n-1} f_i+2\sum_{\substack{i=2\\ i\ even}}^{n-2} f_i\,+\,f_n\Big) \tag{19}$$

where $f_0 = f(a)$ and $f_n = f(b)$. An estimate of the truncation error from the application of the Simpson's rule over the interval between *a* and *b* is

$$E_t = -\frac{(\Delta x)^4}{180}(b\text{-}a)f^{iv}(\overline{x}) \tag{20}$$

where $f''(\overline{x})$ is the average second derivative over the interval. Using equations (19) and (20), we have

$$\int_a^b f(x)dx = \frac{\Delta x}{3}\Big(f_0 + 4\sum_{\substack{i=1\\ i\ odd}}^{n-1} f_i + 2\sum_{\substack{i=2\\ i\ even}}^{n-2} f_i + f_n\Big) - \frac{(\Delta x)^4}{180}(b\text{-}a)f^{iv}(\overline{x}) \tag{21}$$

Simpson's rule is termed a *fourth order* method of numerical integration because the error is proportional to $(\Delta x)^4$.

Example: Demonstrate the use of the Simpson's rule with $n = 4$ to evaluate

$$I = \int_0^\pi \sin(x)dx$$

Simpson's rule with $n = 4$ yields

$$\begin{aligned}
I &= \frac{\pi/4}{3}\{f(0)+4[f(\pi/4)+f(3\pi/4)]+2f(\pi/2)+f(\pi)\}\\
&= \frac{\pi}{12}\{\sin(0)+4[\sin(\pi/4)+\sin(3\pi/4)]+2\sin(\pi/2)+\sin(\pi)\}\\
&= 0.261799[0+4(0.707107+0.707107)+2(1.0)+0]\\
&= 2.00456
\end{aligned}$$