

Management Science

Notes on Transportation Problems

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1 Introduction

Production, storage and consumption points rarely coincide - factories, warehouses and retail outlets are usually geographically separate. Where there are several sources of supply - such as in a multi-plant enterprise - and a number of demand points, the question arises as to which customers are best supplied from which sources. Finding the least cost pattern of shipments is a transportation problem.

Transportation problems turn out to be linear programming problems - but with a very special structure. So much so that it is worth devising a special solution procedure or *algorithm* that takes advantage of this structure. And as even the smallest problem has at least four variables a graphical approach is not possible.

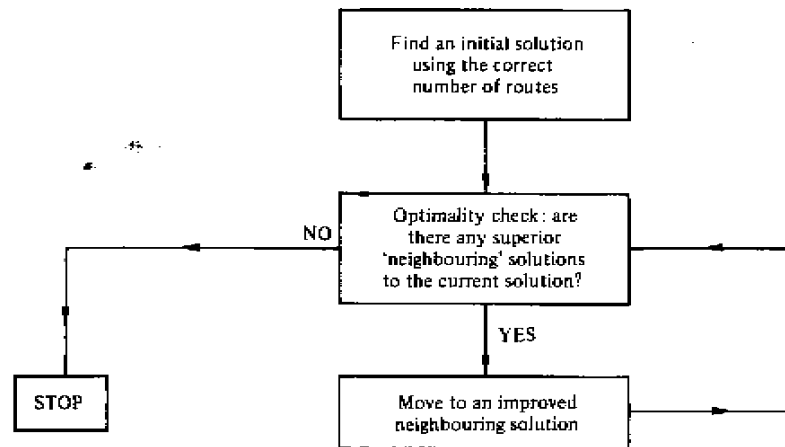
The method also turns out to apply to several problems in other areas of management that have the same underlying structure as a classical transportation problem, but which do not involve the physical movement of goods. One example is *purchasing* (sourcing) problems. Another is short-term financial management - particularly cash which has to be 'transported' from one period of usage to another. In stock control too it is possible to view January's production being 'transported' to July via inventory and the method can also be applied to some problems in personnel selection.

The standard form of the transportation problem was set out by Hitchcock in 1941 and the method of solution employed (sometimes referred to as the 'stepping-stone method') was developed by Charnes and Cooper in 1954.

2 The Logic of the Method

The logic of the solution procedure parallels that of the simplex method. In fact, although it is not obvious (due to a very different tableau arrangement), the iterative phase is a simplified version of the iterative phase of the simplex method, with advantage being taken of the simpler problem structure. Figure 1 sets out the broad flow diagram of the procedure.

Figure 1



In transportation problems, and within the framework of the cost and requirements table, it is possible to find very good initial (starting) solutions with relatively little computational effort. At this point, we shall introduce two methods for finding a good starting solution, although there are many more.

The only solutions that we need to examine - either as starting solutions or in the iterative phase - are those which have the 'correct' number of routes used - i.e. the *basic* feasible solutions. Basic feasible solutions would correspond to corner points in a diagram - if there were enough dimensions to graph the problems.

3 Finding a Starting Solution

Consider the following transportation problem: A company distributing a certain product has three warehouses from which supplies are drawn for four retail customers. The distributor deals in a single product, the supplies of which at each warehouse are as follows:

Warehouse no.	Supply (units)
1	20
2	28
3	17
65 = total supply	

The customer demands are:

Customer no.	Demand (units)
1	15
2	19
3	13
4	18
65 = total demand	

Conveniently, total supply at the warehouses is equal to total demand from the customers. Transportation costs are *per unit* shipped from each warehouse to each customer. Thus it costs

£3 for each unit shipped from warehouse no. 1 to customer no. 1. It costs £5 for each unit shipped from warehouse no. 2 to customer no. 4. The operative word here is *each*. This implies linearity and the absence of any economies or diseconomies of scale.

The data are shown in the form of a *cost and requirements table* with supplies available at each warehouse are shown at the end of each row and the demands from each customer are shown at the foot of each column. Collectively these perimeter data are shown as the *rim conditions* of the problem. When total supply equals total demand the rim conditions are said to be *balanced*.

Warehouse	Customer				Supply
	1	2	3	4	
1	3	6	8	4	20
2	6	1	2	5	28
3	7	8	3	9	17
Demand	15	19	13	18	

All parts of the method require balanced rim conditions but, as we shall see later, a simple adjustment is possible in problems where total supply does not initially equate with total demand.

The first starting solution we shall examine is the commonsense *Minimum Cost Method* (MCM) or *least cost first* (LCF) approach. The MCM procedure is simple starting with the overall cheapest route assign as much as possible to that route and successively make similar *maximal assignments* to the cheapest available remaining route.

So with the minimum cost method, an initial nineteen units are assigned from Warehouse 2 to Customer 2 - here identified as a boldface **19** in the square representing that route. As this completely fills Customer 2's requirement, no further supplies may be shipped there, thus no further entries may be made in column 2. The next cheapest route which is not in column 2 is from Warehouse 2 to Customer 3. The maximum assignment that can be made to this route is the smaller of the remaining supply at Warehouse 2 (9 units) and total demand from Customer 3 (13 units). Hence an entry of **9** is made in square 2,3. No further entries may then be made in the second row.

Of the remaining available squares, square 1,1 and square 3,3 tie for cheapest cost of £3. Resolve this tie in favour of route 1,1 at which a maximal assignment of **15** is possible. The next cheapest route of the four remaining available is from Warehouse 3 to Customer 3 with **4** units of demand unsatisfied. There then remains no option but to place a **5** in square 1,4 and a **13** in square 3,4.

The result is the MCM initial solution:

	Customer				
Warehouse	1	2	3	4	Supply
1	3 15	6	8	4 5	20
2	6	1 19	2 9	5	28
3	7	8	3 4	9 13	17
Demand	15	19	13	18	

The total cost of the Minimum Cost Method solution is found by multiplying the amounts shipped along each route by the cost per unit shipped. This totals £231. Notice that six routes are actually used out of the twelve routes available. Six is the 'correct' number of used routes or 'assignments' - a point we'll return to later.

MCM is an easily obtainable and generally quite good starting solution - these are its main advantages. The main disadvantage of the MCM approach is that, in determining the routes to be used, only one cost figure is considered at a time. How the cost figures stand in relation to each other is important too. This is evidenced by the fact that of the MCM solution's total cost of £231, more than half (£117) resulted from the enforced use of the very expensive route from Warehouse 3 to Customer 4.

Vogel's Approximation Method

VAM takes into account relative costs, provides (on average) a starting solution sufficiently superior to MCM/LCF to warrant the somewhat longer time required to obtain the VAM solution. Indeed, with problems of the size we shall be working with here, the VAM initial solution itself turns out to be optimal in about half the cases.

The VAM starting solution requires the determination of a *VAM number* for each row and column. This number is simply the *difference* in cost between the two best (most economical in minimization) remaining routes in that row or column. The first set of VAM numbers are shown around the perimeter [in ***boldface italic***]:

	1	2	3	4	<i>VAM Nos</i>
1	3	6	8	4	20 <i>1</i>
2	6	1	2	5	28 <i>1</i>
3	7	8	3	9	17 <i>4</i>
	15	19	13	18	
<i>VAM Nos</i>	<i>3</i>	<i>5</i>	<i>1</i>	<i>1</i>	

For example in row 1 the two best routes from Warehouse 1 are to Customer 1 (cost £3) and to Customer 4 (cost £4) thus the VAM difference is £4 - £3 = 1. For column 2 the two best routes to Customer 2 are from warehouse 2 (cost £1) and from Warehouse 1 (cost £6). Thus the VAM number for this column is £6 - £1 = 5. And so on for all the remaining rows and columns.

Now select the row or column with the *largest* VAM number and dispatch as much as possible along the cheapest route in the row or column selected. Thus we select column 2 and send 19 units from Warehouse 2. This is shown as 19 in square 2,2. The maximum dispatch or assignment will be given by the smaller of the supply and demand figures corresponding to the cheapest square in the row or column selected. Thus, Customer 2 is completely supplied and no further entries (for the moment at least) may be made in column 2. This is indicated with crosses in that column. The column is said to be **deleted**. The second set of VAM numbers are now calculated ignoring the deleted column (or row).

	1	2	3	4	VAM Nos		
1	3	X 6	8	4	20	~~1~~	1
2	6	19 1	2	5	28	~~1~~	3
3	7	X 8	3	9	17	4	4
	15	19	13	18			
VAM Nos	~~3~~ 3	~~5~~	~~1~~ 1	~~1~~ 1			

Row 3 now has the largest VAM difference so that 13 (the smaller of 13 and 17) is assigned to the best undeleted square in the row. This is the square 3,3. Once again this deletes a column (it will, of course, not always be a column that is deleted) and VAM numbers are recomputed. The results are:

	1	2	3	4	VAM Nos			
1	3	X 6	X 8	4	20	~~1~~	~~1~~	1
2	6	19 1	X 2	5	28	~~1~~	3	1
3	7	X 8	13 3	9	17	4	4	2
	15	19	13	18				
VAM Nos	~~3~~	~~5~~	~~1~~	~~1~~				
	~~3~~		~~1~~	~~1~~				
	3			1				

Column 1 is now indicated with a VAM difference of 3, so 15 units may be inserted in square 1,1 which deletes column 1. After making this entry, the remaining entries are forced. There *must* be 5 units sent from 1 to 4; there *must* be 4 units dispatched from 3 to 4; and there *must* be 9 units sent from 2 to 4. Only in this way can supplies be exhausted at the warehouses and demand met for Customer 4.

We have now arrived at the VAM *initial solution*. This is:

	1	2	3	4	VAM Nos			
1	3 15	6 X	8 X	4 5	20	~~1~~	~~1~~	1
2	6 X	1 19	2 X	5 9	28	~~1~~	~~3~~	1
3	7 X	8 X	3 13	9 4	17	~~4~~	~~4~~	2
	15	19	13	18				

VAM Nos

3 ————— **5** ————— **1** ————— **1**

3 ————— **1** ————— **1**

3 ————— **1**

The cost of the VAM solution ($15 \times 3 + 5 \times 4$ etc) is £204 which is a significant 11.7% improvement on the MCM solution. VAM gives good results since in contrast to MCM, which simply makes the next assignment to the best square remaining, VAM looks ahead. The VAM number is an *opportunity cost* figure. It is the *least* penalty (at some stage in the problem) per unit that will be incurred if an assignment is *not* placed in the best available square in the row or column concerned. In other words VAM considers some of the *consequences* of choices, whereas MCM does not. Clearly, the MCM solution cannot be optimal here but is VAM? Inspection of the VAM solution above reveals that this is not a trivial question.

4 The Iterative Procedure

We shall use the VAM solution as the starting-point for the iterative phase of the procedure, but note that the solution resulting from the minimum cost method or indeed some other feasible solution with six routes used could serve as a starting solution. But can the VAM solution be improved upon?

Consider first the route Warehouse 2 to Customer 3 which is presently unused but costs only £2 per unit shipped. Would it be worth using this route? In the context of the simplex method, would it be worth bringing a new variable, X_{23} into the solution at unit level? What adjustments would be needed to other shipments to make room for this and what would the cost consequences be?

The necessary changes are shown as italicised figures in the table:

Customer					
Wareh'se	1	2	3	4	Supply
1	3 15	6	8	4 5	20
2	6	1 19	2 +1	5 -1 9	28
3	7	8	3 -1 13	9 +1 4	17
Demand	15	19	13	18	

If +1 is sent from 2 to 3, then one less must be sent from 2 to somewhere else. Less can only be sent to Customers 2 or 4. If less were sent to Customer 2 then more would have to be sent to Customer 2 from somewhere else. This could only be done by introducing another new assignment and it is a key part of the method that only *one* new route is investigated at a time. Solutions which differ in this minimal way correspond to neighbouring corner points and if none of the neighbouring corner points is superior to the current solution then the current solution is optimal.

So, one less must be sent to customer 4 hence the -1 in this square. The remaining changes: one *more* unit to Customer 4 from Warehouse 3 and one less from Warehouse 3 to customer 3, follow from similar arguments. The pattern of changes shown is the *only* pattern of alterations that involves no new assignments other than square 2, 3. The +1 and -1 entries in each row and column must cancel and the changes can be linked by arrows (here shown going clockwise) forming a *closed path* or *closed loop* starting and finishing at 2, 3.

Once a closed path has been formed, the cost effect of the changes can be determined. Adding up the costs at squares where one unit more is being sent ($2 + 9 = 11$) and at places where one unit less will be sent ($3 + 5 = 8$) gives a net *worsening* of £3 in cost. This is represented by $11 - 8 = 3$ which is the actual change in cost and is the *evaluation* of the square 2, 3 in the current solution.

The rules for establishing the closed path for a square are simple. The only 90° turns are allowed, turns can *only* be made at 'assignment' squares (although some used routes may be passed over) the path must finish at the 'vacant' square at which it began (i.e. it must be *closed*). The pattern of adjustments is easily found by putting alternate plus and minus signs at *turning points* in the path, starting with a plus sign at the vacant square being evaluated. The evaluation is then found by adding up costs at 'positive' (increasing level of use) places in the closed path and subtracting the sum of costs at 'negative' (reducing) places in the path. *Negative* evaluations

represent improvement possibility in a minimization problem. Thus square 2, 3 although it is the cheapest unused route, does not at present show improvement.

All the closed paths for the vacant squares in this case are simple rectangular paths for which the following evaluations result:

Square	Evaluation
1, 2	6
1, 3	10

2, 1	2
2, 3	3
3, 1	-1
3, 2	3

Evidently, the only square showing improvement potential is 3, 1 even though it is itself one of the more expensive unused routes. The reason is that using this square enables less to be sent along the even more expensive route 3, 4 which is at a negative place in the closed path. Thus we decide to 'introduce' square 3, 1. Since the problem is a linear programming problem, every unit shipped via 3, 1 will save £1.

The most that can be sent is determined by the *smallest assignment at a negative place in the closed path*. The closed path consists of the four corner squares. Assignment levels will increase at 3, 1 and 1, 4 and decrease at 1, 1 and 3, 4. So when four units have been 'shipped round the path' the assignment in 3, 4 has gone down to zero. This leaves the number of used routes at six and represents a move to an improved basic solution as follows:

Customer					
Wareh'se	1	2	3	4	Supply
1	3 11	6	8	4 9	20
2	6	1 19	2	5 9	28
3	7 4	8	3 13	9	17
Demand	15	19	13	18	

What has been set out so far constitutes a complete solution procedure that need only be repeated to produce the optimum. Closed paths could be established for each vacant square and an evaluation conducted. The closed path for square 2, 3 is a non-rectangular path as may be confirmed but, as always, it is unique. The path for square 3, 2 is also non-rectangular. Closed paths do not *have* to turn when an assignment square is reached (they may need to pass over) and the path may cross itself. There is no significance in either of these events.

Although we shall still need closed paths, there is a quicker way of carrying out the vacant-square evaluations. Numbers (call them u_i) are assigned to each row and numbers (call them v_j) are attached to each column in such a way that if square i, j is an assignment square, then $u_i + v_j = \text{cost for that square}$.

It is necessary to determine one of these values arbitrarily and the convention is to set $u_1 = 0$ (as an exercise, you could check that it makes no difference to the subsequent *evaluations* of cells if, say, v_2 was first set = 7 or $u_3 = -22$). Now, if $u_1 = 0$ and, by the rule $u_i + v_l = \text{cost in square } i, l$ (since there is an assignment here) then $u_1 + v_l = 3$ so that $v_l = 3$. Next, since there is also an assignment in square 1, 4 then $u_1 + v_4 = \text{cost in square } 1, 4 = 4$ so that $v_4 = 4$. Note that there is no need for $u_1 + v_2 = 6$ since there is no assignment in square 1, 2. Continuing, if $v_4 = 4$ then $v_4 + u_2 = 5$, so $u_2 = 1$. If $u_2 = 1$ and $u_2 + v_2 = 1$, $v_2 = 0$. Now, we found earlier that $v_l = 3$. If $v_l = 3$ and $u_3 + v_l = 7$, then $u_3 = 4$. If $u_3 = 4$ and $u_3 + v_3 = 3$, then $v_3 = -1$. This completes the row and column numbering:

	v1 = 3	v2 = 0	v3 = -1	v4 = 4	Supply
u1 = 0	3 11	6	8	4 9	20
u2 = 1	6	1 19	2	5 9	28
u3 = 4	7 4	8	3 13	9	17
Demand	15	19	13	18	

Negative numbers are allowed although most will be positive. There is no connection between u and v numbers (*fictitious costs*) and VAM numbers. Vacant squares are now quickly evaluated. The evaluation of any square i, j shown as e_{ij} is:

$$e_{ij} = c_{ij} - (u_i + v_j)$$

so that the following results are produced:

Square	u_i	v_j	c_{ij}	Evaluation
1, 2	0	0	6	6
1, 3	0	-1	8	9
2, 1	1	3	6	2
2, 3	1	-1	2	2
3, 2	4	0	8	4
3, 4	4	4	9	1

These evaluations may be checked by the closed-path method. All evaluations are positive, which indicates that the solution is optimal and unique (no zero evaluations other than for the occupied cells) with a total cost of £200. Had it *not* been optimal, we should have picked the square with the largest negative evaluation, established the closed path for the square, shipped as much round that path as possible, recalculated new u_i and v_j (they change when the solution does) and re-evaluated vacant squares. The procedure stops when there are no negative evaluations. If a vacant square has a zero evaluation this means that there is another solution with the same cost; the solution would be *non-unique*. As an exercise, use the u_i and v_j method to check the evaluations obtained for the VAM initial solution.

The use of the u_i and v_j row and column numbers and closed paths has been called the *modified distribution method* (MODI) the *row-column sum method* or the *stepping-stone method* is a highly efficient solution procedure based upon (although not superficially resembling) the simplex method for Linear Programming. Indeed it is sometimes also called the *transportation simplex method*.

This brings us to the point of explaining why six assignments is the correct number for the current problem. Since transportation problems are Linear Programming problems of special structure, only basic feasible solutions need to be considered. In a transportation problem with m rows and n columns all basic feasible solutions have $m + n - 1$ routes used - hence six in the current problem. It is no accident that the MCM solution and the VAM solution gave the correct number of assignments. This is because at each stage a row or column-fulfilling assignment (the maximum possible) was made. There would therefore be $m + n$ assignments or routes used were it not for the fact that the last assignment made will simultaneously fulfil a row *and* a

column because total supply equals total demand. The modified distribution method then preserves this number of assignments by introducing only one new route at a time to the maximum possible extent resulting in the elimination of one of the existing routes.

5 Sensitivity Analysis: The Objective Function

Consider first the objective function coefficients. As with LP as a whole, any *proportionate* increase or decrease in all objective function coefficients will leave the optimum solution unchanged. Route usages should be the same and total cost will go up or down in the given proportion. So, for example, uniform inflation of costs will not change the optimal pattern of distribution. In transportation problems it is also the case that an equal *absolute* amount added to all objective function coefficients in any row or column or throughout the entire problem will leave the optimum arrangement unchanged.

Suppose now that a change occurs to an individual cost. Individual cost data may change when, for example, a new road is built or a weight restriction introduced on part of a route. If the changed cost is that of an unused route, any increase cannot bring the route into consideration - the evaluation e_{ij} will worsen. If, however, the cost reduces so that e_{ij} for an unused route becomes negative, then the original solution is no longer optimal and the route in question should be introduced. The limit for the original solution to remain optimal is, for an unused route, a reduction in cost of e_{ij} . Thus for square 2, 1 with an original cost figure of 6, the original solution will remain optimal so long as the cost in that square c_{21} is such that:

$$4 \leq c_{21} < \infty$$

This is the tolerance interval for c_{21} .

If the cost of a used route changes, this will affect some of the u_i and v_j values. The original solution will remain optimal so long as none of the evaluations that depend on the changed u_i and v_j have become negative. Consider an example. Let the cost in square 3, 1 be c_{31} instead of 7. The u_i and v_j values are then:

	$v_1 = 3$	$v_2 = 0$	$6 - c_{31}$	$v_4 = 4$	Supply
$u_1 = 0$	3 11	6	8	4 9	20
$u_2 = 1$	6	1 19	2	5 9	28
$c_{31} - 3$	7 4	8	3 13	9	17
Demand	15	19	13	18	

The only evaluations that change are for routes in column 3 and row 3: e_{13} , e_{23} , e_{32} and e_{34} . These are recalculated in terms of C_{31} which must not be such as the cause the e_{ij} to become positive. The results are:

$$\begin{aligned}
 e_{13} &= 8 - (0 + 6 - C_{31}) \geq 0 & \text{so } C_{31} &\geq -2 \\
 e_{23} &= 2 - (1 + 6 - C_{31}) \geq 0 & \text{so } C_{31} &\geq 5 \\
 e_{32} &= 8 - (C_{31} - 3 + 0) \geq 0 & \text{so } C_{31} &\leq 11 \\
 e_{34} &= 9 - (C_{31} - 3 + 4) \geq 0 & \text{so } C_{31} &\leq 8
 \end{aligned}$$

All of these bounds must be observed or at least one of the e_{ij} will become positive. The tolerance interval for c_{31} is therefore between the greatest of the lower bounds (GLB) and the least of the upper bounds (LUB) thus:

$$5 \leq c_{31} \leq 8$$

Similar analysis can be carried out on each objective function coefficient individually.

6 Sensitivity Analysis: The Right Hand Side

Now consider changes in the right hand side (RHS) elements - i.e. the supply and demand figures (also known as the *rim conditions*). We can determine the effects of changed rim conditions by use of the u_j and v_i which are *dual values*, but in the transportation problem only the relative sizes of the u_i and v_j are meaningful.

To illustrate the use of the u_i and v_j , consider the following situation. Management wishes to run down Warehouse 3 while making a compensating increase in capacity at Warehouse 1. How much would this save? It is easy to see that this would save £4 for every unit of capacity so transferred. This would be effected by one unit less from Warehouse 3 to Customer 1 (saving £7) and one unit more from Warehouse 1 to Customer 1 (costing an extra £3). The same saving of £4 is arrived at by multiplying the u values by the respective changes in supply. The change in cost, ΔC is given by:

$$\Delta C = u_3(-1) + u_1(+1) = -4$$

This figure would hold up to a limit of four units of capacity transferred by which time use of route 3, 1 would have been reduced to zero. In a similar way, the result of a unit of capacity transferred from Warehouse 3 to Warehouse 2 can be determined. The result is:

$$\Delta C = u_3(-1) + u_2(+1) = -3$$

In a similar way, balanced changes of demand figures can be checked for their effect on costs. One unit less demand from Customer 1 combined with one unit more from Customer 3 would save £4:

$$\Delta C = v_1(-1) + v_3(+1) = -4$$

Simultaneous and balanced changes in supply and demand figures can also be evaluated. If there were 2 units more demanded from Customer 1 with the extra 2 units of supply available at Warehouse 2 the effect on cost would be an increase of £8:

$$\Delta C = v_1(+2) + u_2(+2) = 8$$

The cost consequences of a varied pattern of changes in supplies and demands can be worked out by multiplying each change by the u_i or v_j figure in its row or column and summing up. This will give the effect on costs subject to two conditions:

- 1 That overall supply remains equal to overall demand.
- 2 That the changes are such that only the original routes continue to be used.

In respect of condition 1 the method can be extended to deal with this situation. In respect of 2 the tableau must be inspected to verify this. Several adjustments may be needed to the assignment levels. A simple gross check is that if the supply and demand adjustments in total

are not in excess of the smallest assignment in the tableau it *must* be possible to accommodate the changes using the same routes.

Consider now what would happen if one more unit was demanded by Customer 3 with the extra units supply at Warehouse 1. Total cost would actually go *down* by £1 as a result of the *extra* unit to be shipped. This phenomenon is called the *paradox of transportation* or the 'more for less' phenomenon.

The extra demand is met by shipping one more unit along each of routes 1, 1 and 3, 3 and one less along route 3, 1. Cost can go down overall because the changed pattern of supply and demand allows more intensive use of highly efficient routes (1, 1 and 3, 3) with less need to use a very expensive one (3,1).

Finally, in respect of the u_i and v_j , note that if all of the u_i and v_j values are multiplied by their corresponding supply and demand figures and summed the result is:

$$0(20) + 1(28) + 4(17) + 3(15) + 0(19) - 1(13) + 4(18) = 200$$

which is the minimum total cost figure. The u_i and v_j are dual values and 200 is the value of the objective function of the dual problem at the optimum. This equality provides a useful gross check on the workings.

7 Unequal Supply and Demand

The transportation simplex method (also known as MODI) works for problems expressed in terms of balanced rim conditions. Where demand does not equal supply at the outset, a simple device can be used to restore the arithmetic balance.

In the transportation problem shown below, the supply at the warehouses totals 104 while demand in total is only 93.

	1	2	3	4	Supply
A	8	13	9	7	33
B	15	13	11	16	41
C	13	8	12	11	30
Demand	25	20	26	22	

To make up the difference, an additional column, the fifth in this problem and referred to as a *dummy*, is inserted with the 'missing' or balancing demand. All dummy costs are zero. Any warehouse that ships to a dummy has that many units left in stock. In this case (only the optimum is shown here) it should be warehouse B.

Dummy

	V1 = 8	v2 = 4	v3 = 4	v4 = 7	v = -7	Supply
u1 = 0	8 21	13	9	7 12	0	33
u2 = 7	15 4	13	11 26	16	0 11	41
u3 = 4	13	8 20	12	11 10	0	30
Demand	25	20	26	22	11	

If demand exceeded supply, a balancing, *dummy row* would be inserted. Whichever customers then received 'supplies' from the imaginary source would be short on their orders. In obtaining the VAM initial solution and in all subsequent workings, the dummy column or row and its 'costs' are treated in just the same way as all other rows and columns.

8 Prohibited Routes

Not all sources of supply may have routes available to all customers and sometimes it may be necessary to ensure that entries are not made in certain cells in dummy rows and columns. These are examples of prohibited routes. For such routes, a cost figure of M is attached to the route. The unspecified amount M is large enough to dominate any other number with which it is compared - as, for example, in vacant square evaluations. Thus, for example $-0.1M + 3,000$ is a negative number and $-5,000 + 0.01M$ is a positive number. With M cost entries made where necessary the workings - both VAM and the iterative phase - proceed as before.

Consider the problem above. Suppose that the view of company management was that it was essential that supply at Depot B is exhausted. This means that Depot B to the dummy customer is a prohibited route and a cost of M instead of zero should be entered in this square. The new optimal solution will then be:

	V1 = 8	v2 = 6	v3 = 4	v4 = 7	Dummy v = -2	Supply
u1 = 0	8 11	13	9	7 22	0	33
u2 = 7	15 14	13 1	11 26	16	0	41
u3 = 7	13	8 19	12	11	0 11	30
Demand	25	20	26	22	11	

The cost for the solution is £903, which can be compared with the £868 cost of the previous optimum. The difference in cost, £35, is the price that must be paid to fulfil the provision that Warehouse B supply be used up. This is valuable management information. A price can always be put on a particular provision by solving the problem with and without that provision and taking the difference.

9 Degeneracy

A difficulty that sometimes occurs in transportation problems is *degeneracy*. A degenerate solution has too few assignments to establish closed paths for all vacant squares. Equivalently not all U_j and v_j can be determined. A degenerate solution is shown below:

	1	2	3	Supply
A	8	10 15	7	15
B	6 8	10	9	8
C	10 2	8	4 20	22
Demand	10	15	20	

The only square for which a closed path can be found is B3. The degeneracy is resolved by adding in a zero assignment - effectively to form an artificially occupied cell. The zero assignment can be placed at any vacant square for which a closed path cannot be established. Such a cell is called an *independent location*. In this case only B3 is *not* independent. In the table below, the zero is placed in square A1.

	$v_1 = 8$	$v_2 = 10$	$v_3 = 2$	Supply
$u_1 = 0$	8 0	10 15	7	15
$u_2 = -2$	6 8	10	9	8
$u_3 = 2$	10 2	8	4 20	22
Demand	10	15	20	

There is no fundamental reason for this choice over another independent location other than, as a tie-breaker, the use of the first independently located cell. With the artificially occupied cell, the u_i and v_j can be found and the optimum follows in one iteration:

	$v_1 = 8$	$v_2 = 10$	$v_3 = 6$	Supply
$u_1 = 0$	8 2	10 13	7	15
$u_2 = -2$	6 8	10	9	8
$u_3 = -2$	10	8 2	4 20	22
Demand	10	15	20	

Note that the optimal solution is not degenerate.

Sometimes a choice of location results only in the zero being relocated. One of two things must then result. *Either* a position for the zero can be found where a real amount may be shipped round a worthwhile path *or* where there are no negative vacant square evaluations (i.e. the solution was already optimal although not recognizably so at first).

10 Optimal Sourcing

The transportation model can be useful in problems that don't involve physical movement of goods, or where movement of goods is incidental. As an example, consider the following problem in procurement. A department store requires the following annual quantities of five types of clothing items:

Type	A	B	C	D	E
Quantity required (k)	18	9	12	20	16

The management of the store has quotations from four manufacturers who have undertaken to supply not more than the quantities below (of all types combined).

Manufacturer	1	2	3	4
Can Supply (k)	20	18	25	19

The store's management accountants estimate that its profits per item will vary with the type and manufacturer as follows:

	A	B	C	D	E
1	2.0	1.9	2.3	1.5	3.2
2	1.8	1.9	2.1	1.6	2.8
3	2.5	2.4	2.2	1.7	3.6
4	2.2	1.4	2.1	1.8	2.8

The store management wants to know the optimal sourcing arrangement that would produce maximum profit and whether any discretion is available in the form of equally profitable alternative arrangements.

As an additional consideration, suppose that the store was already contracted to purchase 8,000 units of type C from manufacturer 2 (these 8,000) units being included in the demand for C and the maximum supply from 2 given above). What is the maximum amount it ought to be prepared to pay to be released from this contract?

The distinctive feature here is that the problem is one of *maximization*. In terms of Vogel's Approximation Method, the VAM numbers will now be the difference between the two *best* squares. So for row 1, the first VAM number will be:

$$3.2 - 2.3 = 0.9$$

The VAM initial solution is shown below and in which it will be seen that a dummy column was needed and that the solution is degenerate. The degeneracy has been resolved by location of a zero in square 1, E.

	A	B	C	D	E	Dummy	
	v1 = 2	v2 = 2	v3=2.3	V4=1.6	v5=3.2	v6 = 0	
u1 = 0	2.0 8	1.9	2.3 12	1.5	3.2 0	0	20
u2 = 0	1.8	1.9	2.1	1.6 11	2.8	0 7	18
u3=0.4	2.5	2.4 9	2.2	1.7	3.6 16	0	25
u4=0.2	2.2 10	1.4	2.1	1.8 9	2.8	0	19
	18	9	12	20	16	7	

The solution produced by VAM is not optimal. Even though the problem is one of maximization, the u_j and v_j numbers can be determined as before, viz., $u_j + v_j = \text{profit in square } i, j$. The difference is that, as regards the evaluations, it is *positive* evaluations that now represent improvement.

Thus for square 1, B the evaluation is:

$$\text{profit}(1, 2) - (u_1 + v_2) = 1.9 - (0 + 2) = -0.1$$

which means that less profit will be made (Note: you check this by establishing the closed path, and bear in mind that the method does not 'know' what the financial entries in the cells represent).

Square 3, A has a positive evaluation of:

$$\text{profit}(3, 1) - (u_3 + v_1) = 2.5 - (0.4 + 2) = +0.1$$

and eight (thousand) units can be shipped round the closed path corresponding to this square. Doing this - as it happens - removes the degeneracy and the solution that is produced is optimal. The solution is:

	A	B	C	D	E	Dummy	
	v1=2.1	v2 = 2	v3=2.3	V4=1.7	v5=3.2	v6=0.1	
u1 = 0	2.0	1.9	2.3 12	1.5	3.2 8	0	20
u2=-0.1	1.8	1.9	2.1	1.6 11	2.8	0 7	18
u3=0.4	2.5 8	2.4 9	2.2	1.7	3.6 8	0	25
u4=0.1	2.2 10	1.4	2.1	1.8 9	2.8	0	19
	18	9	12	20	16	7	

The total profit made is £179,400. The solution, however, is *non-unique*. Square 2, B has a zero evaluation and the solution resulting from shipment around the corresponding closed path will produce an equally good arrangement so discretion is available.

As regards any existing contract to purchase 8,000 units of type C from manufacturer 2, this would change profit by:

$$£8(1,000)(2.1 - (-0.1 + 2.3)) = -£800$$

That is, there would be a reduction of £800. The result of keeping to the contract would be to make an assignment of 8 in square 2, C, so that if the release costs less than £800 it would be worthwhile.

11 Additional Topics

A wide range of situations can be modelled by making appropriate changes to the basic cost and requirements layout but retaining its essential structure and efficient solution procedures. For example, in *transshipment* problems there are intermediate locations through which goods may pass.

In *capacitated* problems there are limits on the amounts that may be sent along some or all routes. There may also be minimal usage necessary along some routes (or minimal purchases from some sources).

Diseconomies of scale can be allowed for (a discrete increase in unit cost along a route if used above a certain level). Unfortunately, *economies of scale*, which are perhaps more frequent, are more troublesome.

Some models allow for 'losses in transit' - for instance, electrical power losses in transmission or deterioration in storage. The US Navy used this kind of model to minimize transportation and overhaul costs at naval shipyards.

In some logistical problems all that matters is that total distribution is completed in the shortest possible time. This defines a *least time* transportation problem.

Other areas of application of the transportation model include short-term investment planning and the management of cash.

Other starting solutions include *Russell's Approximation Method* (of comparable efficiency to VAM) and *Houthakker's Method* of mutually preferred flows.

It should by this stage be evident both that the range of application of transportation problems is considerable, the solution procedure is highly efficient and that the basic framework is very adaptable.