The Binomial distribution

The binomial distribution is a theoretical (regular) discrete probability distribution that is mainly used to calculate probabilities in experiments where there can either be

1. Two possible outcomes or
2. Several outcomes classified into two groups.

For example, the binomial distribution can be used in a coin-tossing experiment since only two outcomes are considered – heads and tails (the possibility that the coin falls on its edge is not deemed considerable given the rarity of the event). The application of the binomial distribution may also be considered in a die-throwing experiment whereby, for example, the outcomes may be classified as 'even' and 'odd' numbers. Usually, one outcome (group of outcomes) is labelled as *success* and the other as *failure*.

Conditions for using the binomial distribution

More essentially, the choice of the binomial distribution rests on the satisfaction of two main conditions:

1. The trials in the experiment must be mutually independent and
2. The probability of obtaining a success for each trial must remain constant throughout the entire experiment.

The first condition simply means that the outcome of a trial should in no way affect the outcome of a subsequent trial. The second condition may be explained as follows – if the probability of obtaining 'heads' when tossing a coin is one-third, it should remain one-third till the end of the experiment, that, for every toss (or trial).

Note

The binomial distribution can be also be used when selecting *with replacement* or selecting *without replacement from an infinite population*. If items are selection is made with replacement from a bag containing, for example, 10 red and 5 blue balls, then the probability of obtaining a red ball at every selection will always remain two-thirds. Alternatively, even though an item is not replaced after selection from an infinite population, the probability of a success differs by so little that it is considered to have remained constant.

Objective of the binomial distribution

The binomial distribution is mainly used to calculate the probability of obtaining r successes in n trials. The distribution is completely characterised by its two parameters n (total number of trials) and p, the probability of obtaining a success on each trial. The notation used for a variable X following a binomial distribution is $X \sim \text{Bin}(n, p)$, X being defined as the 'number of successes'. Let us illustrate the application of the binomial distribution by means of a very simple example.

Example

Evaluate the probability of obtaining exactly three sixes in ten throws of an ordinary fair die.

Solution

The 'ordinary fair' die is in fact *six-sided* and *unbiased* so that the probability of obtaining any score (from 1 to 6) on any throw is $\frac{1}{6}$. If we consider that obtaining a six is a success, then we can define X = "number of sixes". The number of trials is n = 10 and the probability of obtaining a six is $\frac{1}{6}$; we are required to calculate P[X = 3] given that $X \sim$ Bin(10, $\frac{1}{6}$).

In order to find the answer, we need to know the *probability mass function* of the binomial distribution, that is, the formula that generates the probabilities corresponding to all the values assumed by X.

The situation is identical to putting 3 letters in 10 pigeonholes where any pigeonhole cannot contain more than one letter at a time. It means that we either have an empty pigeonhole (failure) or a pigeonhole containing a letter (success).

The following is a schematic representation of *one successful combination*, that is, 3 letters spread over 10 pigeonholes:

1	2	3	4	5	6	7	8	9	10
X				X		X			
$\frac{1}{6}$	$\frac{5}{6}$	$\frac{5}{6}$	$\frac{5}{6}$	$\frac{1}{6}$	$\frac{5}{6}$	$\frac{1}{6}$	$\frac{5}{6}$	$\frac{5}{6}$	$\frac{5}{6}$

Each box marked with a cross (X) indicates that a 6 has been obtained (1^{st}, 5^{th} and 7^{th} throws). An empty box indicates that a number other than 6, that is, 1, 2, 3, 4 or 5, has been obtained. Note that the probability of obtaining a 6 (success) for each trial is $\frac{1}{6}$ and that the probability of not obtaining a 6 is $\frac{5}{6}$. We deduce that the probability of a successful combination occurring is the product of all the probabilities in the above table (since the throws are independent) and that is equal to $\left(\frac{1}{6}\right)^3\left(\frac{5}{6}\right)^7$. The number of such successful combinations is ${}^{10}C_3$, similar to choosing 3 boxes out of 10 without any specific order. Since all successful combinations have the same probability of occurring, P[X = 3] = ${}^{10}C_3\left(\frac{1}{6}\right)^3\left(\frac{5}{6}\right)^7$.

Note

The above discussion can be extrapolated to a general case where the number of trials is n, the probability of a success on each trial is p and the probability required being P[X = r]. It can easily be deduced that the probability mass function of the binomial distribution is given by

$$P[X = r] = {}^{n}C_r\, p^r (1-p)^{n-r} \text{ for r} = 0, 1, 2, \ldots, n$$